The Duration of Sovereign Default

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Abstract

The observed sovereign default spells are lengthy on average, with large variability in duration, both across countries and over time. Moreover, in the data, the distribution of the durations of default is highly skewed and looks exponential. We show that a model of dynamic contracting with private information, calibrated to the international data on sovereign default, can quantitatively account for these observations. Our model also suggests that the larger output variability that international lending supports may explain the observation that the average duration of default is more than two times longer in the thirty years after than before 1980.

JEL: D86, F34, H63

Keywords: sovereign lending, default duration, dynamic contracting

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1 Introduction

The hallmark of the market for sovereign debt is that defaults occur periodically with individual sovereign countries. Reinhart and Rogoff (2011) call this the "serial default", a widespread phenomenon especially across emerging markets. These defaults may occur five or fifty years apart; they may be wholesale default or a partial default through rescheduling.[1] During any default, the defaulting country is either completely excluded from the world credit market, or it must face extremely high interest rates for new loans. Moreover, the observed default spells are lengthy on average, with large variability in duration, both across countries and over time. From Reinhart and Rogoff (2011), among the 52 countries in their dataset that experienced at least one default episode over the period 1800-2010, the average number of years the sovereign county is in default per default episode ranges from 1 to 41 years, with an average of 9.6 years.

More interestingly, as Uribe and Schmitt-Grohé (2017) find, the distribution of the length of default is highly skewed and looks exponential. This is shown in Figure 1, which presents the observed distribution of default durations from the same dataset of Reinhart and Rogoff (2011). Uribe and Schmitt-Grohé (2017) also discover is that the average duration of default varies between differential sample periods. They find, for example, average default duration in the period 1975–2014 is shorter than that in 1824–2014. Our own calculation shows that over the more recent time after WWII, default durations are on average more than two times longer in the 30 years after 1980 than before.

Mainstream theories of sovereign default model international lending in incomplete markets.[2] Lending is carried out in standard debt contracts that specify a constant repayment, and the sovereign country is not able to commit to its debt obligations. If the current state of the world is such that the net gains from defaulting dominates that of not defaulting, the country will choose to default. In these models, it is typically assumed that once default occurs, the sovereign country will face in each period a constant probability to re-enter the world credit market.

These models generate quantitative outcomes that resemble observed default cycles, while, because of the constant probability of regaining access to international credit, automatically

[1] Tomz and Wright (2013), who document the occurrence of sovereign debt default in history, find "The most frequent defaulters were Ecuador, Mexico, Uruguay, and Venezuela; each experienced at least 8 distinct spells of default, exemplifying a phenomenon Reinhart and Rogoff (2004) call "serial default." Ecuador and Honduras have each spent more than 120 years in default, beginning with their initial loans as members of the Central American Confederation in the 1820s, and Greece has been in default for more than 90 years of our sample."

[2] This includes Eaton and Gersovitz (1981); Arellano (2008); Mendoza and Yue (2012).

producing an exponential distribution of default durations. What's not adequate with these models, however, is that they usually do not give an account for the economic mechanism behind the constant probability of regaining access to international financial markets. In these model, it's not clear what ends the default or what determines the duration of the default once it starts. They, therefore, could not be viewed as a theory for the observed exponential distribution of default durations; neither should they be expected to provide an account for why the durations were longer in thirty years after than prior 1980.

In this paper, we use a model of dynamic contracting – a variation of Luo and Wang (2015) – to quantitatively account for the data on default durations, as well as a set of other characteristics that define observed sovereign lending relationships. In any period, if the sovereign country's investment is funded internally, it produces a low but constant autarkic output; externally, its output is stochastic but higher on average. Output is privately observed. An optimal contract is designed to determine whether lending should occur in a given period, and to provide incentives for the sovereign country to repay its debt. In this environment, default is interpreted as a state of the dynamic contract where the sovereign country ceases, completely or partially, to repay the credit of the international lender; the lender suspends the borrower's access to international lending; and the parties enter a new continuation of the contract where the values of both the borrower and the lender are significantly marked down.

In the model, default (i.e., suspension in lending) occurs because the low output has been reported too many times, and default allows a penalty, imposed on the sovereign country, to be implemented. After a default episode ends, lending then reemerges with a "restructured" contract where incentives are reorganized to support the next cycle of financial lending. The duration of the individual default episode, which is chosen optimally as part of the contract, depends on the size of the penalty that the lender needs to impose on the sovereign country. It also depends on the willingness of the sovereign country to continue to repay its debt in the state of default.

The model is constructed to capture an essential feature of the observed sovereign lending relationship: that the lender must depend on the borrower or the sovereign nation's willingness, not just its ability, to repay the debt. In the model, the lender cannot impose bankruptcy on the borrower – to seize the ownership or replace the management of his assets. What the lender can do is to terminate the lending, either temporarily or permanently, as such an action arises optimally from his perspective.

To replicate the observed distribution of the duration of default, an important variable is the sovereign country's willingness to repay its debt – modeled specifically as a minimum level of the sovereign country's consumption which bounds its ability to repay debts.

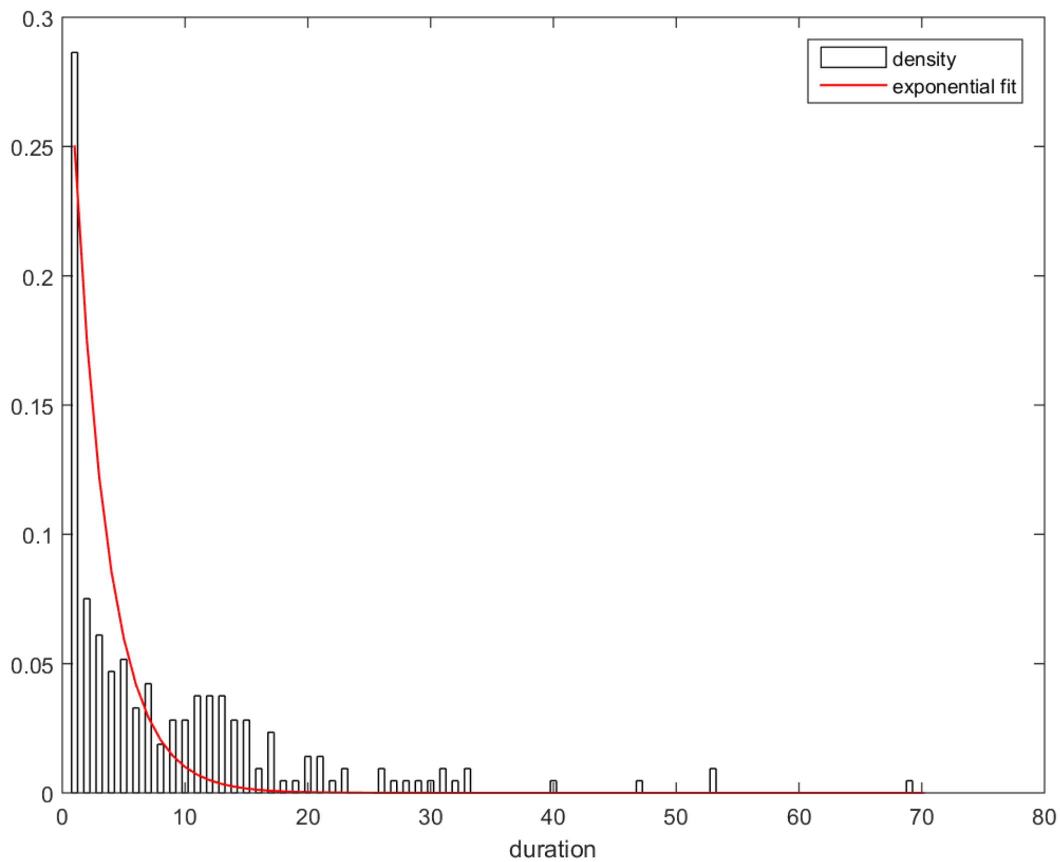


Figure 1: Distribution of duration

Source: Own calculations based on the dataset of Reinhart and Rogoff (2011) which includes 70 countries, over 1800-2010.

Note: This figure depicts the observed density of the distribution of the length of default. Defaults longer than 40 years are not included.

1.1 The literature

As noted earlier, the literature offers ample empirical evidence showing that observed durations of sovereign defaults are lengthy on average and with large variability (Richmond and Dias, 2009; Trebesch, 2010; Reinhart and Rogoff, 2011; Benjamin and Wright, 2013). Benjamin and Wright (2013), for example, show that the average default takes more than 8 years to resolve, results in creditor losses of roughly 50 percent, and leaves the sovereign country as or more highly indebted than when they entered default. Their dataset also shows great variability in the length of the default spell across countries and over time.

Richmond and Dias (2009) study empirically the factors that determine the duration of the exclusion of the defaulting nation from international capital markets between 1980–2005. They find that partial or full market access depends mostly on external demand for risk, good domestic behavior and market expectations. The defaulter's size also matters, with large economies regaining market access twice as fast as small countries. Recent theories view the process of default resolution as a dynamic bargaining game where the duration of the default is determined as part of the game's outcome (Bi, 2008; Bai and Zhang, 2012; Pitchford and Wright, 2012; Benjamin and Wright, 2013). For example, in Pitchford and Wright (2012), since the sovereign nation cannot commit to making identical settlement offers to all creditors, delay arises endogenously because creditors have incentives wait for better terms of settlement at a later date. With a similar argument, Pitchford and Wright (2016) show the distribution of delay for the entire settlement process is a weighted sum of gamma distributions. Bai and Zhang (2012) observe that over the period 1990–2005, sovereign debt renegotiations take an average of five years for bank loans but only one year for bonds. They argue that information revelation in the secondary market plays a crucial role in shortening debt renegotiations. Renegotiations of bank loans take longer to complete relative to bonds because bank loans are rarely traded while bonds are heavily traded on the secondary market.

The rest of the paper is organized as follows. Section 2 presents the observations on the duration of sovereign default. Section 3 describes the model. Section 4 formulates the problem of optimal contracting and characterizes the optimal contract. Section 5 calibrates the model to the U.S. data. Section 6 concludes the paper.

2 Sovereign Default Durations

How robust is the observation of the exponential duration distribution? Divide the data of Reinhart and Rogoff (2011) into two sub-sample periods, prior and after WWII, with the duration distributions for the two sub-samples shown in Figures 2 and 3, respectively. For each sub-sample, the distribution looks exponential, as the for entire sample. The data of Reinhart and Rogoff (2011) is divided into smaller and larger sub-samples, in many different ways we experimented, the resulting default durations all look exponential.

Given the exponential observed distributions of default durations, we then ask: do the moments of the distribution, mean in particular, stay constant over time across sub-samples of the data? The answer, already given by Uribe and Schmitt-Grohé (2017), is negative. Observe that durations are on average more than two times longer prior to 1945 than after 1945. Observe, more interestingly, that over the sixty years between 1961-2010, durations in the second half of the period are on average three times as long as those in the second half of the period.

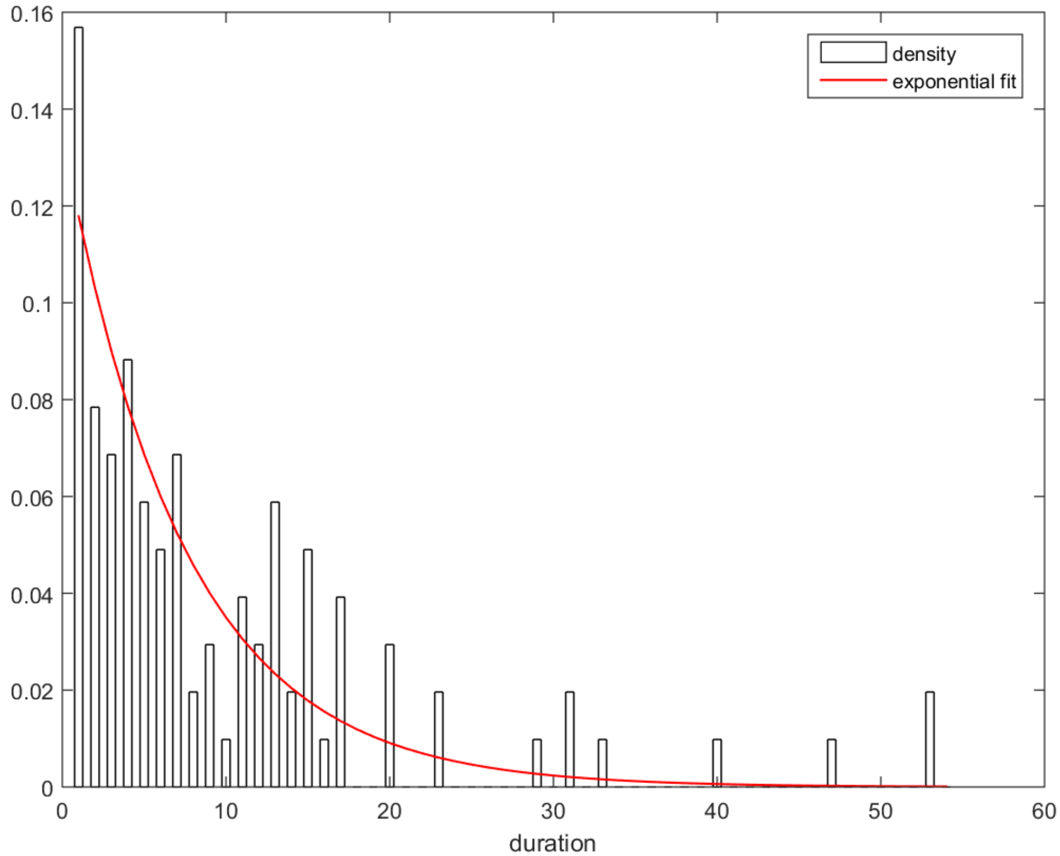


Figure 2: Distribution of duration over 1800 - 1945

Source: Own calculations based on the dataset of Reinhart and Rogoff (2011) which includes 70 countries. Defaults longer than 40 years are not included.

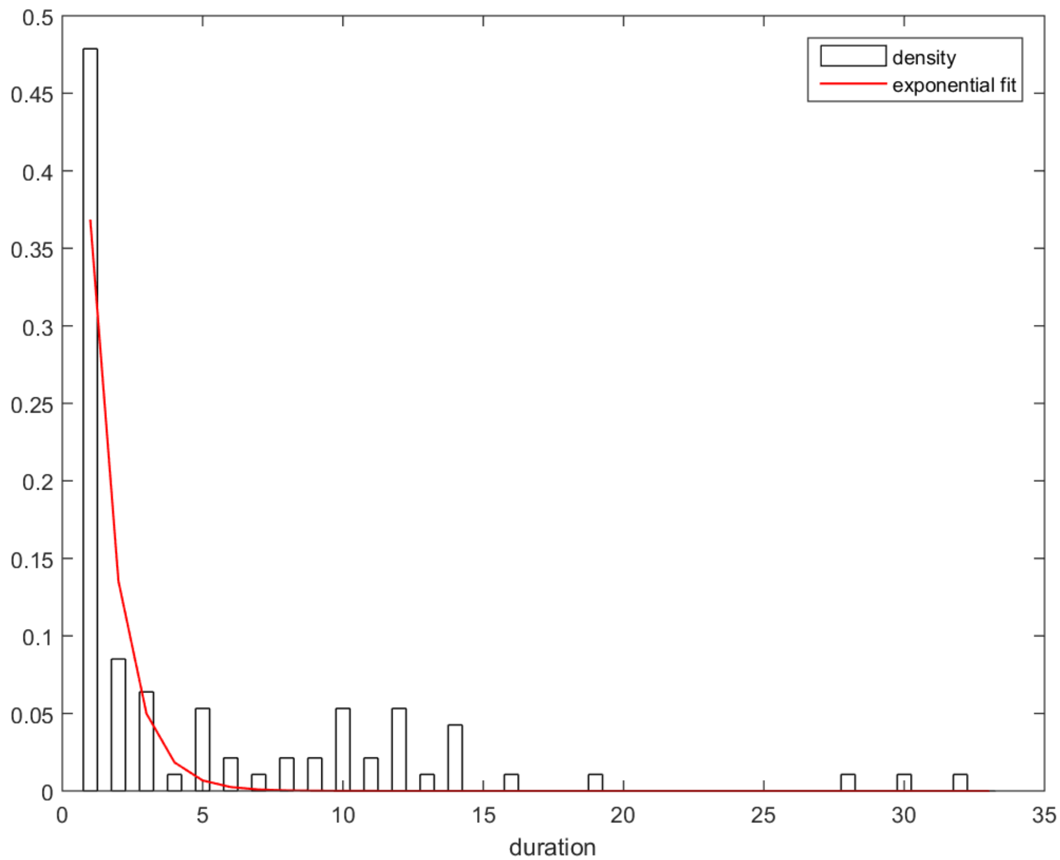


Figure 3: Distribution of duration over 1946 - 2010

Source: Own calculations based on the dataset of Reinhart and Rogoff (2011) which includes 70 countries. Defaults longer than 40 years are not included.

Following Uribe and Schmitt-Grohé (2017), we calculate the frequency and the length of sovereign defaults for the various sample periods selected, as presented in Table 1. Specifically,

$$\text{The probability of default per year} = \frac{D}{N \times T}, \tag{1}$$

where D is the number of default episodes within the period; N is the number of countries who defaulted at least once during the period; T is the number of years. And

$$\text{Average duration per default episode} = \sum_{1}^{D} d_i/D, \tag{2}$$

where d_i is the duration of default episode i.

Table 1: Frequency and average duration of sovereign defaults

Period	Probability of default per year	Average duration per default episode
1800 - 1945	0.02	9.99
1946 - 2010	0.03	5.18
1951 - 1980	0.07	1.85
1981 - 2010	0.05	6.52

Source: Own calculations based on the dataset of Reinhart and Rogoff (2011) which includes 70 countries.

Our goal in this paper is to account for these observations, using a simple model which we now present.

3 Model

Let t denote time: $t = 1, 2, \cdots$ There is a single perishable consumption good in the model. There are two agents, a lender (international lending institution, creditor) and a borrower (the sovereign nation, debtor), both infinitely alive. The lender is risk neutral and maximizes expected lifetime returns from lending. The borrower is risk averse and maximizes

$$\mathbb{E}_\tau \sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t) \ ,$$

where $\mathbb{E}_\tau$ denotes his expectation conditional on information available at the beginning of period τ, $\tau \geq 1$; $\beta \in (0, 1)$ is the discount factor which he shares with the lender; c_t and $u(c_t)$ denote, respectively, the borrower's consumption and utility in period t. Assume the utility function u is bounded, strictly increasing, strictly concave, twice differentiable. In addition, $-u''(c)/u'(c)$ is non-increasing in c.

The borrower owns a project that can run in one of two states: in the state of no lending – the autarkic state – in which the borrower fails to obtain external finance, it produces, using domestically available capital in a fixed amount which is normalized to zero, a known and constant return θ_0; and in the state of lending, where a fixed amount of capital K, which the project requires and the lender provides, is invested in the project, it returns a stochastic output $\theta \in \{\theta_1, \theta_2\}$, with $\pi_i \in (0, 1)$ being the probability with which $\theta = \theta_i = \tilde{\theta}_i \theta_0$, $i = 1, 2$, and $\pi_1 + \pi_2 = 1$. Assume $0 \leq \theta_0 < \theta_1 - K < \theta_2 - K$. That is, lending makes the project more productive, in both the low and high output states.

As in Luo and Wang (2015), the availability of international lending puts the sovereign country on a superior technological track. With the use of foreign capital, not only is the sovereign country able to produce more efficiently, with higher mean in output, but also the output produced entails greater volatility and informational complexity.

Capital, internally provided or externally financed, is perishable. Any period the project must operate in its autarkic state if the required external capital is not available. In any state of lending, however, the realization of the stochastic output of the project is observed by the borrower who runs the project, but not the lender who provides the capital.

In this environment, the lender and the borrower develop a contract to support lending, subject to three constraints:

First, the lender is committed to the terms of any contract he wishes to enter, but the borrower is free to quit the contract at the beginning or end of each period if continuing the contract no longer offers a value greater his value of autarky – the value he gets if he produces the autarkic output θ_0 for the rest of his life.[3]

Second, the lender has limited ability in controlling the borrower's assets. Although the lender is free to make decisions in each period about whether or not to provide finance for the borrower, he is *not* able in any period to take control over the borrower's project. More specifically, the lender could not either seize the debtor's assets (the project), or to impose on him to hand over the control of the project to any third party.

Third, consumption in the sovereign country must satisfy $c_t \geq \underline{c}$ for all t, where $\underline{c} \in (0, \theta_0)$ is a constant. That is, consumption must be above a given minimum level for the sovereign country. The idea is that this minimum level in consumption defines an aspect of the sovereign country's willingness to repay its debt, especially in the state of default. In particular, it imposes that the debt repayment in any period of default cannot exceed $\theta_0 - \underline{c}$.

It has been emphasized in the literature that the sovereign country's willingness to repay the deft is important for resolving the default.This will turn out to be the case in our model. In the quantitative version of the model, this will be important for generating the distribution of default durations that matches data.

[3]The notion of limited commitment can be more generally modeled. We may assume, for example, that the borrower could not promise to stay in the relationship if he is promised a value below some given V_o which denotes his outside option, with $V_o \geq V_{\min}$. When $V_o > V_{\min}$, the story would be that the borrower is free to take, in each period, an outside offer from a third party lender who promises V_o. We could also allow the borrower to leave the contract in the middle, not just the beginning and end, of any period after the random output is realized in that period. This would slightly complicate the analysis but not change qualitatively the main predictions of the model.

4 Optimal Contracting

For any period, the contract must prescribe whether lending should occur in that period. We allow the contract to randomize between lending and suspension. Following the literature, we use the borrower's beginning-of-period expected utility, denoted V, as a state variable to summarize his current history with the lender. A dynamic lending contract, formulated recursively, then takes the form of

$$\sigma \equiv \{I(V), m_0(V), V_0(V), m_1(V), V_1(V), m_2(V), V_2(V) : V \in \Sigma\},$$

where Σ is the state space – the set of all expected utilities of the borrower, the Vs, that a feasible contract in this environment can achieve. This set is an endogenous variable of the model. Then for all $V \in \Sigma$, $I(V)$ is the probability of suspension – the probability with which lending is in suspension in the current period; and $1 - I(V)$ is the probability of lending – the probability with which lending takes place in the current period. Again for all $V \in \Sigma$, $m_i(V)\,(i = 0, 1, 2)$ is the borrower's payment to the lender if his current output is θ_i (or $\theta_i - m_i(V)$ is his current consumption); and $V_i(V)$ is the borrower's expected utility at the beginning of the next period if his current output is θ_i, $i = 0, 1, 2$.

Let $V_{\min} \equiv u(\theta_0)/(1-\beta)$ and $V_{\max} \equiv u(+\infty)/(1-\beta)$ denote, respectively, the borrower's expected utility in autarky and the *sup* of the expected utility he could achieve. Obviously, $\Sigma \subseteq [V_{\min}, V_{\max})$. Next, for each $V \in \Sigma$, let $U(V)$ denote the lender's maximum value attainable through a feasible and incentive compatible contract. Then the value function $U(\cdot)$ must be the solution to the following Bellman equation, called problem $\mathbf{P}$: $\forall V \in \Sigma$,

$$U(V) = \max_{I,\{m_i,V_i\}_{i=0,1,2}} I(m_0 + \beta U(V_0)) + (1-I)\left(\sum_{i=1}^{2} \pi_i(m_i + \beta U(V_i)) - K\right)$$

subject to

$$I(u(\theta_0 - m_0) + \beta V_0) + (1-I)\left(\sum_{i=1}^{2} \pi_i(u(\theta_i - m_i) + \beta V_i)\right) = V, \tag{3}$$

$$u(\theta_2 - m_2) + \beta V_2 \geq u(\theta_2 - m_1) + \beta V_1, \tag{4}$$

$$u(\theta_1 - m_1) + \beta V_1 \geq u(\theta_1 - m_2) + \beta V_2, \tag{5}$$

$$m_i \leq \theta_i - \underline{c}, \quad i = 0, 1, 2, \tag{6}$$

$$V_i \in \Sigma, \quad i = 0, 1, 2, \tag{7}$$

$$\Sigma \subseteq [V_{\min}, V_{\max}), \tag{8}$$

$$I \in [0,1], \tag{9}$$

and that Σ is the largest self-generating set with respect to (3)-(9).

In the above, equation (3) is the promise-keeping constraint which requires that the values of the current choices be consistent with the definition of V. Equations (4) and (5) are incentive compatibility – the borrower has incentives to truthfully report his current output. Equation (6) is a limited liability constraint which requires that the repayment from the borrower not exceed the difference between the current output and his minimum consumption $\underline{c}$. Equation (7) requires that the expected utility promised to the borrower be feasible for the contract to deliver. Equation (8) is a self-enforcing constraint which requires that the expected utility promised to the borrower be greater than his autarkic value. Last, Σ being the largest self-generating set with respect to (3)-(9) requires that the space of the state variable V be the largest consistent with the above constraints.

As in Luo and Wang (2015), $\Sigma = [V_{\min}, V_{\max})$. In other words, the least expected utility a feasible and incentive compatible contract can deliver for the borrower is his autarkic value, and any higher level of expected utility the physical environment permits is attainable with a feasible and incentive compatible contract.

The above optimization problem can be divided, as we show in Appendix A, into three semi-independent sub-problems. The first sub-problem decides, for each V, whether lending should occur in the current period. The second decides, conditional on suspension (i.e., lending does not occur in the current period), what actions and payoffs are optimal in the current period. Finally, the third sub-problem solves for the optimal actions and payoffs for the parties conditional on lending occurring in the current period. Specifically, problem **P** can be restated as

$$\mathbf{PI}: \forall V \in \Sigma: \; U(V) = \max_{\{I,V_s,V_l\}} IU_s(V_s) + (1-I)U_l(V_l)$$

subject to (9) and

$$IV_s + (1-I)V_l = V,$$

$$V_s \in [V_{\min}, V_{\max}), V_l \in [\widetilde{V}, V_{\max}),$$

where $U_s(\cdot)$, which defines the lender's values conditional on suspension, is given by

$$\mathbf{PS}: \quad \forall V_s \in \Sigma: \; U_s(V_s) = \max_{\{m_0,V_0\}} m_0 + \beta U(V_0)$$

subject to

$$u(\theta_0 - m_0) + \beta V_0 = V_s, \tag{10}$$

$$m_0 \leq \theta_0 - \underline{c}, \tag{11}$$

$$V_0 \in [V_{\min}, V_{\max}), \tag{12}$$

and $U_l(\cdot)$, which defines the lender's values conditional on lending, is given by

$$\mathbf{PL}: \quad \forall V_l \in [\widetilde{V}, V_{\max}) : U_l(V_l) = \max_{\{m_i, V_i\}_{i=1,2}} \sum_{i=1}^{2} \pi_i \left(m_i + \beta U(V_i)\right) - K$$

subject to (4), (5) and

$$\begin{aligned} \sum_{i=1}^{2} \pi_i \left(u(\theta_i - m_i) + \beta V_i\right) &= V_l, \\ m_i \leq \theta_i - \underline{c}, \ i &= 1, 2, \\ V_i \in [V_{\min}, V_{\max}), \ i &= 1, 2, \\ \widetilde{V} \equiv \sum_{i=1}^{2} \pi_i u(\theta_i - \theta_1 + \underline{c}) &+ \beta V_{\min}, \end{aligned} \tag{13}$$

where $\widetilde{V}$ being the minimum level of expected utility of the borrower that is feasible for the contract to attain, conditional on lending.

To see why $\widetilde{V}$ is the minimum attainable expected utility that supports lending, notice that conditional on lending, given output is privately observed, it is feasible for the borrower to report θ_1 in both of the output states and obtain an expected utility of

$$\sum_{i=1}^{2} \pi_i u(\theta_i - m_1) + \beta V_1 \geq \sum_{i=1}^{2} \pi_i u(\theta_i - \theta_1 + \underline{c}) + \beta V_{\min} = \widetilde{V}.$$

So any expected utility that the contract promises to the borrower must be at least $\widetilde{V}$. Moreover, it can be shown, by way of construction, that any $V \in [\widetilde{V}, V_{\max})$ can be attained with a feasible and incentive compatible contract. In other words, the domain for problem **PL** is indeed $[\widetilde{V}, V_{\max})$. Notice that $\widetilde{V}$ is strictly increasing in $\underline{c}$.

Let $\widehat{V} = u(\underline{c}) + \beta\widetilde{V}$. Following Luo and Wang (2015), we have

Theorem 1. *(i) Suppose*

$$0 \leq V_{\min} \equiv \frac{u(\theta_0)}{1-\beta} < \frac{\beta[\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})]}{1-\beta^2}.$$

Then the optimal contract has

$$I(V) = \begin{cases} 1 & \text{if } V \in [V_{\min}, \widehat{V}] \\ (\widetilde{V} - V)/(\widetilde{V} - \widehat{V}) & \text{if } V \in (\widehat{V}, \widetilde{V}) \\ 0 & \text{if } V \in [\widetilde{V}, V_{\max}) \end{cases} ; \tag{14}$$

(ii) Suppose

$$\frac{\beta[\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})]}{1 - \beta^2} \leq V_{\min} < \frac{\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})}{1 - \beta}. \tag{15}$$

Then the optimal contract has

$$I(V) = \begin{cases} 1 & \text{if } V = V_{\min} \\ (\widetilde{V} - V)/(\widetilde{V} - V_{\min}) & \text{if } V \in (V_{\min}, \widetilde{V}) \\ 0 & \text{if } V \in [\widetilde{V}, V_{\max}) \end{cases} ,$$

(iii) Suppose

$$V_{\min} \geq \frac{\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})}{1 - \beta}. \tag{16}$$

Then the optimal contract has $I(V) = 0, \ \forall V \in [V_{\min}, V_{\max})$.

(iv) Over the states of suspension, the optimal contract has

$$V_0(V_s) = \frac{V_s - u(\underline{c})}{\beta} > V_s, \quad m_0(V_s) = \theta_0 - \underline{c}, \tag{17}$$

and over the states of lending,

$$V_1(V_l) < V_l < V_2(V_l), \quad m_1(V_l) < m_2(V_l). \tag{18}$$

The main difference between this model and that of Luo and Wang (2015) is constraints (11) and (13), which impose a lower bound on the borrower's consumption in each output state. In the appendix, we show that whether or not these constraints bind, they do not affect the characterizations of the optimal contract. Specifically, conditional on suspension, suppose (11) binds. Then for any $V_s \in [V_{\min}, V_{\max})$ we have $V_0(V_s) = [V_s - u(\underline{c})]/\beta > V_s$. Similarly, conditional on lending, suppose (13) binds. Then for any $V_l \in [\widetilde{V}, V_{\max})$, $V_2(V_l) = (V_l + \pi_1 u(\theta_2 - \theta_1 + \underline{c}) - (1 + \pi_1)u(\underline{c}))/\beta > V_l$.

Remember $\widetilde{V}$ is the minimum level of the borrower's expected utility above which lending can be supported and below which not. Equation (14) thus indicates that if lending can be supported in the current period, it never pays to wait until a future period to start/restart it. Equation (14) also says that for V sufficiently small, below $\widehat{V}$ specifically, it is optimal to suspend lending in the current period. For V that is neither sufficiently small, $V > \widetilde{V}$, nor sufficiently large, $V < \widetilde{V}$, it is optimal to mix between suspension at $\widehat{V}$ and lending at $\widetilde{V}$.

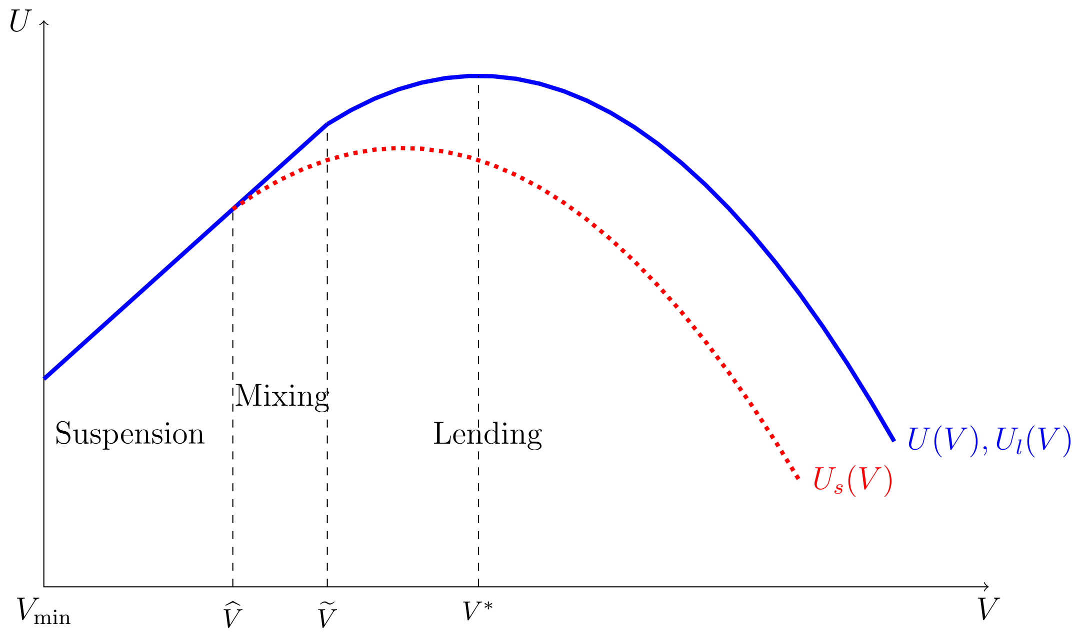


Figure 4: The lender's value functions, $U(\cdot)$, $U_l(\cdot)$, $U_s(\cdot)$.

Over any episode of default, the optimal contract would move the borrower up in expected utility (i.e., , $V_0(V) > V$) while at the same time requiring him to submit to the lender the current output after subtracting his minimum consumption (i.e., $m_0(V) = \theta_0 - \bar{c}$). Conditional on lending, the borrower's current debt repayment and his expected utility from next period on are increasing in the current output he produces and reports.

Figure 4 illustrates the lender's value functions and the associated optimal lending/suspension policies. Observe that $U(\cdot)$ has a constant and positive slope over the interval $[V_{\min}, \widetilde{V})$ on which a positive probability of suspension is prescribed. Figure 5 illustrates the optimal law of motion of the borrower's expected utility V, that equations (17) and (18) describes.

Obviously, it is the lower output produced in the states of suspension relative to that in the states of lending that gives rise to the positive slope in the lender's value function over the interval $[V_{\min}, \widetilde{V})$. Conditional on suspension or partial suspension (i.e., over the interval $[V_{\min}, \widetilde{V})$), a lower V moves the relationship farther away from lending, reducing the lender's

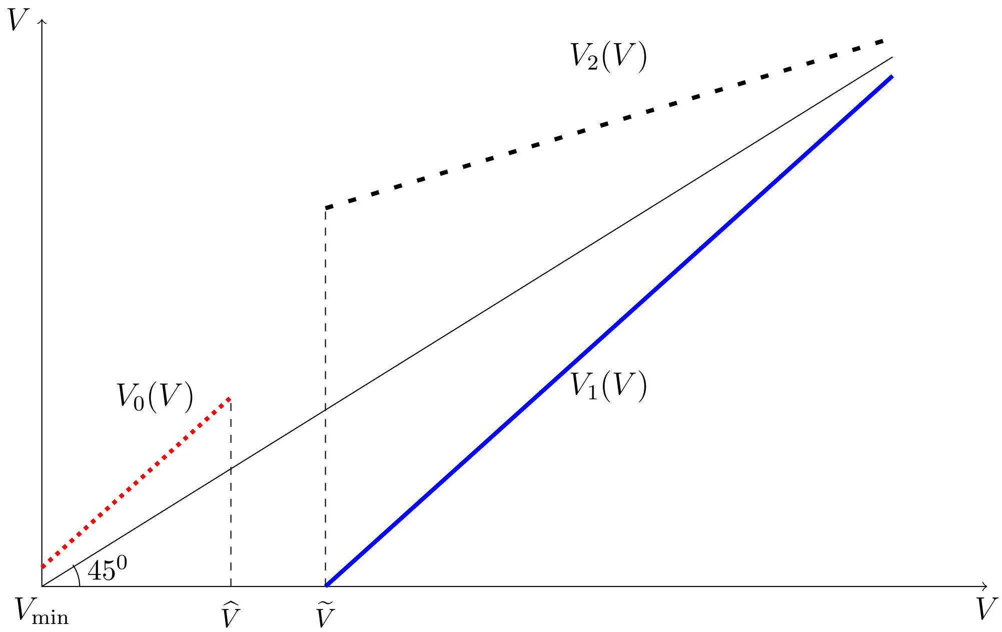


Figure 5: Law of motion for the borrower's expected utility.

value. Conditional on lending (i.e., for $V \geq \widetilde{V}$), a lower V moves the relationship closer to suspension, exerting again a negative effect on the lender's value. These effects then work, simultaneously but in an opposite direction, with the usual effect that higher levels of V reduce the lender's value by requiring more compensation to the borrower, to give rise to the shape of the value function shown in Figure 4.[4]

Theorem 1 suggests that how much suspension – interpreted as a state of default – that the optimal contract prescribes depends on how high the sovereign country's autarkic output is, or how strong the demand is for lending. Specifically, for θ_0 or $V_{\min}$ sufficiently low, the optimal contract prescribes, on the equilibrium path and depending on the borrower's expected utility, both deterministic and randomized suspension. For θ_0 larger, the optimal contract prescribes only randomized suspension. Last, for θ_0 sufficiently large, so large that $V_{\min} \geq \widetilde{V}$,[5] then all V greater than $V_{\min}$ can support lending, and the optimal contract prescribes no suspension at all.

A larger θ_0 increases $V_{\min}$, the minimum borrower's expected utility the contract could enforce in a state of suspension (and in all states of the optimal contract). In other words, the larger autarkic value reduces the maximum penalty the lender can impose on the borrower,

[4]Standard models of dynamic contracting with private information and a binding limited liability (or non-negativity in compensation) constraint could also generate a positive slope of the principal's value. That effect is not in this model.

[5]The holds if and only if $V_{\min} \geq [\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})]/(1-\beta)$.

by way of putting the lending on suspension. This lowers the efficiency of suspension as an incentive device. The increased $V_{\min}$ in turns increases $\widetilde{V}$. The more severe penalties not being feasible, lending now requires higher expected utilities for the borrower (the lower expected utilities which initially support lending are no longer consistent with promise-keeping). Thus a larger θ_0 shifts the interval $[V_{\min}, \widetilde{V})$ – the set of expected utilities associated with suspension – to the right. More importantly, the measure of $[V_{\min}, \widetilde{V})$ decreases from being positive to negative, as $V_{\min}$ increases from 0 to be above $\widetilde{V}$, moving the optimal contract in the direction of less suspension, as Propositions 1 suggests.

There is another channel through which θ_0 affects incentives and suspension. Notice that θ_0 not only determines the borrower's outside value, it is also the maximum (and in fact the optimal) debt repayment in the states of suspension. A higher θ_0 makes suspension less costly and should therefore encourage the lender to use suspension more for incentives. Obviously, however, this effect is dominated by the effect discussed above at least for θ_0 sufficiently large.[6]

From the same propositions, there is also an indication that the optimal use of suspension increases as the lender's demand for incentives conditional on lending, which should increase in $\theta_2 - \theta_1$, increases. Note that a larger $\theta_2 - \theta_1$ increases the debtor's gains from misreporting θ_2 as θ_1, making the incentive problem more severe. Notice that a larger $\theta_2 - \theta_1$ pushes up the two cutoffs in the propositions, expanding the support for suspension and squeezing the space for lending.

Consider next the effects of a higher $\underline{c}$ – a lower willingness to repay – on default and its duration. Notice first that a higher $\underline{c}$ implies a higher $\tilde{V}$. This expands the set $[V_{\min}, \tilde{V})$ – the states with which default occurs with a positive probability. Notice next from equation (17) that the sovereign country starts a episode of default from $V = V_{\min}$, then a larger $\underline{c}$ implies a longer duration of default – the distance the sovereign country must travel, $\tilde{V} - V_{\min}$ is longer, and each step it takes to end the default, measured by

$$V_0(V) - V = \frac{(1-\beta)V - u(\underline{c})}{\beta},$$

is small. (Notice also that a larger β implies a smaller step up each period.)

[6] One way to separate the two effects is to consider an extension of the model where θ_0 is still the borrower's autarkic output, but his outside value is independently given, denoted V_o for example. Then an increase in θ_0 would increase suspension but a higher V_o should reduce suspension. In an earlier version of the paper, we show in a numerical example that a larger θ_0 induces the optimal contract to push the borrower faster from a given state of lending into suspension, while a lower θ_0 gives the contract stronger incentives to stay above $\widetilde{V}$ – away from suspension.

5 Quantitative Analysis

In this section, we take the model to the data, letting i denote an individual country in a collection of countries in the dataset. We first calibrate/estimate the model to obtain values of the model's parameters. we then use these parameter values to simulate the model and it show that it can produce observed country distributions of default durations.

5.1 Data

The data on default we use in the estimation is taken from Reinhart and Rogoff (2011), which contains information on episodes of external default in a collection of 70 countries over the period 1800-2010. The data on the real GDP per capita is from the Maddison Project (Bolt and van Zanden, 2014). We choose to focus on the more recent post-World War II period 1951-2010, which is then divided into two sub-periods, which are 1951-1980 and 1980-2010. We normalize the average real GDP per capita over 1951-1980 to be 1.

For each sub-period, we first exclude default events that start before the sub-period begins or end after the sub-period ends. Next, to estimate θ_{0i}, $\tilde{\theta}_1$ and $\tilde{\theta}_2$, we exclude the countries that never defaulted during the sub-period, or had only one non-default episode. Last, to make sense for the cross-country comparison and the computation of the counter-facturals, we only consider the countries that are not excluded in both sub-periods, which gives us a total of 12 countries, as shown in Table 2.

5.2 Calibration

To calibrate the model, we assume the sovereign country has an CRRA utility function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},\ \forall c \geq 0,$$

where we set $\sigma = 2$ as is common in business cycle models. The discount factor β is set to be 0.96 given that the model will be calibrated to annual data. We assume that the the country's minimum consumption is a fraction of the autarkic output, $\underline{c}_i = \lambda\theta_{0i}$, and we set $\lambda = 0.99$ as a benchmark.

To calibrate the production technology, we assume the countries differ in their autarkic GDP, θ_{i0}, but not in how international capital improves on the autarkic output. Specificaly, assume, for each individual country i, its output in any non-default period is given by:

$$\theta_i = \tilde{\theta}\theta_{0i}, \tag{19}$$

where

$$\tilde{\theta} = \begin{cases} \tilde{\theta}_1, & \text{with probability } \pi \\ \tilde{\theta}_2, & \text{with probability } 1-\pi \end{cases},$$

where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are constants, with $0 < \tilde{\theta}_1 < \tilde{\theta}_2$. That is, international lending affects the sovereign country's output *multiplicatively*, and through a random variable $\tilde{\theta}$ which is i.i.d across time and countries.

The above structure, by way of putting a restriction on how the model looks at the cross-country gains from international lending, greatly reduces the dimensionality of the calibration/estimation of the production functions, from potentially $4N$ to $N+3$, N being the number of countries in the sample. Note, again, that the random variable $\tilde{\theta}$ measures the gains in productivity that international lending brings about. These gains are stochastic but constant across individual countries.

5.2.1 Estimating π, θ_{0i}, $\tilde{\theta}_1$ and $\tilde{\theta}_2$

For each individual country i, the estimate of its autarkic output θ_{0i} is given by the average GDP of the country over the default periods, denoted $\hat{\theta}_{0i}$. The remaining parameters of the model, π, $\tilde{\theta}_1$ and $\tilde{\theta}_2$, are then estimated, in the following, using a specific version of GMM.

The theoretical mean of the non-default GDP of country i is

$$(\pi\tilde{\theta}_1 + (1-\pi)\tilde{\theta}_2)\theta_{0i},$$

and the theoretical standard deviation of the non-default GDP of country i is

$$\sqrt{\pi(1-\pi)}(\tilde{\theta}_2 - \tilde{\theta}_1)\theta_{0i}.$$

Let $\xi = (\pi, \tilde{\theta}_1, \tilde{\theta}_2)$. Let $\mathbf{m}(\xi) = (\mathbf{m}_1(\xi)\ \mathbf{m}_2(\xi)\ \cdots\ \mathbf{m}_N(\xi))^T$, where for each i,

$$\mathbf{m}_i(\xi) = \begin{pmatrix} \widehat{E(\tilde{\theta}_i)} - (\pi\tilde{\theta}_1 + (1-\pi)\tilde{\theta}_2) \\ \widehat{SD(\tilde{\theta}_i)} - \sqrt{\pi(1-\pi)}(\tilde{\theta}_2 - \tilde{\theta}_1) \end{pmatrix},$$

where $\widehat{E(\tilde{\theta}_i)}$ and $\widehat{SD(\tilde{\theta}_i)}$ are, respectively, the sample mean and sample standard deviation of country i's non-default GDP over it autarkic output, or θ_i/θ_{0i}. Specifically,

$$\widehat{E(\tilde{\theta}_i)} = \frac{1}{T_i}\sum_{t=1}^{T_i}\frac{\theta_{i,t}}{\hat{\theta}_{0i}},$$

and

$$\widehat{SD(\tilde{\theta}_i)} = \left[\frac{1}{T_i} \sum_{t=1}^{T_i} \left(\frac{\theta_{i,t}}{\hat{\theta}_{0i}} - \widehat{E(\tilde{\theta}_i)} \right)^2 \right]^{1/2},$$

where for each i, $\theta_{i,t}$ is country i's non-default GDP in the sample. And we estimate $\xi = (\pi, \tilde{\theta}_1, \tilde{\theta}_2)$ by choosing the values of π, $\tilde{\theta}_1$ and $\tilde{\theta}_2$) to minimize the value of $\mathbf{m}(\xi)'\mathbf{m}(\xi)$.[7]

For making the desired comparison, we estimate the parameters using data from two separate sample periods, 1981-2010 and 1951-1980, respectively,

The estimated values of θ_{0i} are shown in Table 2. Observe the great cross country variability in average output in the states of default (the estimated θ_{0i}), both before and after 1980. Observe also that among the countries in our dataset, most of the countries showed substantial development in their own ability to produce, while others going in the opposite direction.[8]

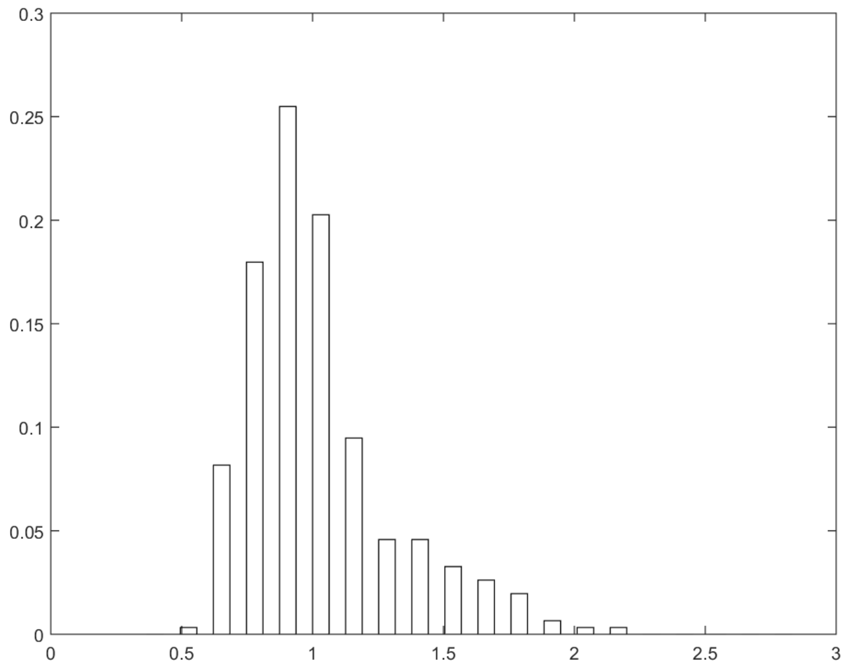
[7]More generally, we could minimize $\mathbf{m}(\xi)'\mathbf{W^{-1}}\mathbf{m}(\xi)$, where $\mathbf{W^{-1}}$, the weighting matrix, chosen optimally, should be the inverse of the variance-covariance matrix of the simulated moments. Here, for convenience we use the identity matrix instead, which is not optimal but does make the estimate consistent.

[8]This includes, most notably, Zimbabwe, Ghana, Peru, and Chile.

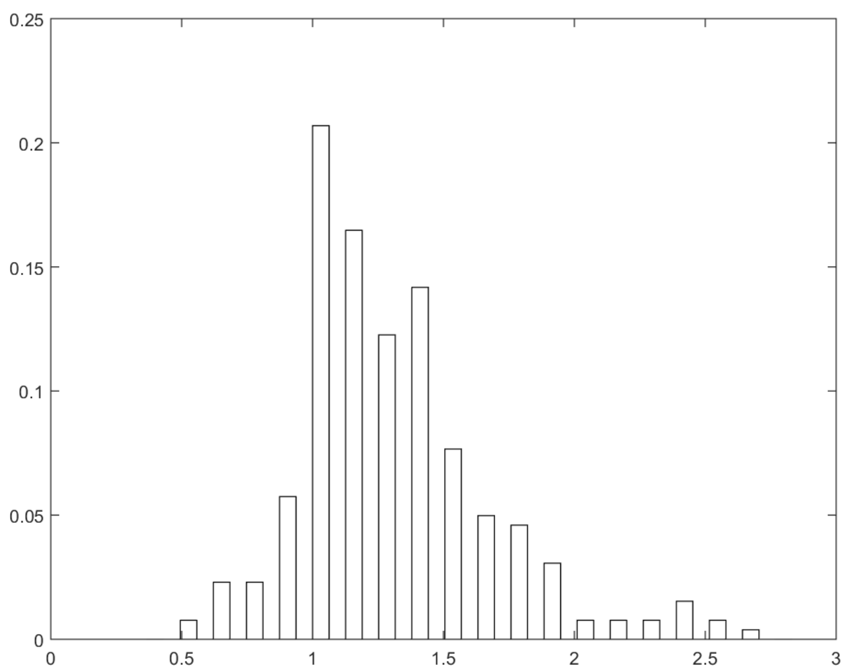
Table 2: The estimated autarkic output θ_{0i}

Country	1951-1980		1981-2010	
	θ_{0i}	Normalized value	θ_{0i}	Normalized value
Zimbabwe	1196.80	0.41	928.10	0.32
Ghana	1387.75	0.48	1007.00	0.35
Sri Lanka	1766.00	0.61	1980.00	0.68
Paraguay	1799.50	0.62	3144.89	1.08
Indonesia	1050.60	0.36	3226.50	1.11
Peru	4064.75	1.40	3486.29	1.20
Costa Rica	2785.00	0.96	4518.33	1.55
Brazil	2454.50	0.84	5004.54	1.72
Turkey	3286.00	1.13	5108.50	1.75
Chile	4712.67	1.62	5502.75	1.89
Uruguay	4860.00	1.67	6209.71	2.13
Argentina	5600.73	1.92	7326.00	2.51

Note: The unit in the table is the 1990 Geary Khamis dollar, based on PPP.



(a) The distribution in 1951-1980



(b) The distribution in 1981-2010

Figure 6: Country productivities that international lending supports

Figure 6 depicts, for the two sample periods we have chosen to look at respectively, the total "sample" of $\tilde{\theta}$, where each sample point is a value of $\tilde{\theta}_i = \theta_i/\hat{\theta}_{0i}$, where $\hat{\theta}_{0i}$ is the estimate of θ_{0i} just obtained. Note, of course, that these values are calculated across all countries but only for the non-default periods. These samples will be used in the following for estimating the values of π, $\tilde{\theta}_1$ and $\tilde{\theta}_2$.

Notice the shift to the right in the sample distribution after 1980, which indicates that, overall, international lending has been more effective in increasing the sovereign country's productivity over its autarky.

Table 3: Estimated $\tilde{\theta}_1$ and $\tilde{\theta}_2$

	1951-1980	1981-2010
π	0.9	0.6
$\tilde{\theta}_1$	1.1	1.1
$\tilde{\theta}_2$	1.6	1.6
$\mathrm{E}(\tilde{\theta})$	1.15	1.3
$\mathrm{VAR}(\tilde{\theta})$	0.02	0.06

The estimated values of π, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are then repotrted in Table 3 which shows, from the perspective of our simple model, significant differences in the productivity gains that international lending helps achieve between the two sample periods we look at. Notice that the differences result mainly from the lowered probability of the low output, π. Between 1951-1980 and 1981-2010, relative to the country's autarkic technology, international lending brings both higher mean and higher variance in GDP.[9]

5.3 Simulation

In this section, we simulate the model using the parameter values obtained in the prior section. For each sub-period and with the estimated parameter values for the sub-period, we first run the model for each individual country to compute the optimal contract. We then use the optimal contract to produce dynamics on lending and default over a period of 150 years. We then select the default episodes between the periods $101 - 130$ for calculating model generated average default duration for the individual country.[10] Note that in doing this we follow the

[9]This is consistent with existing empirical findings.
[10]In this way, each country starts randomly at period 101.

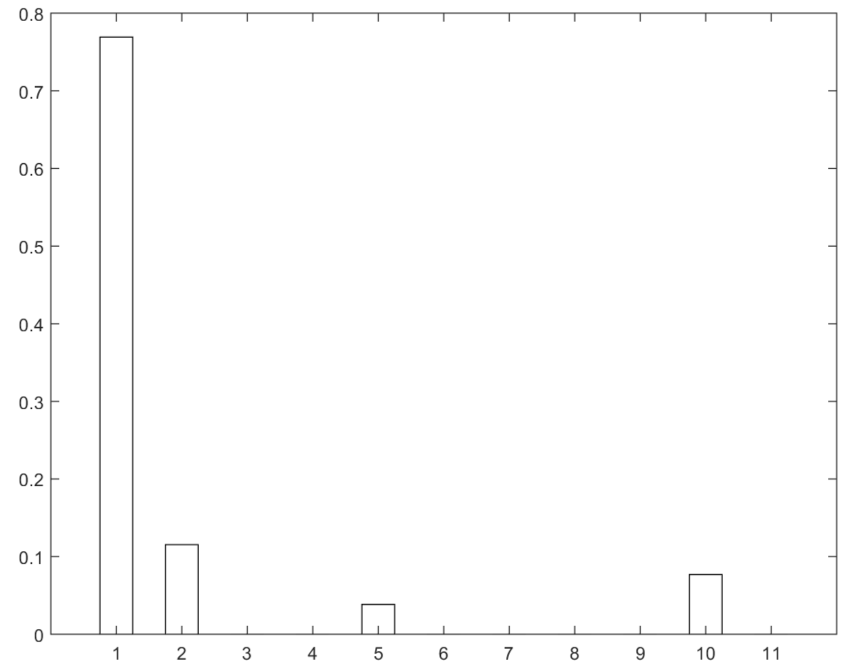
rules used for identifying an individual episode in the data. Specifically, we excludes default episodes that start before 101 or ends after 130. We then aggregate across all individual countries to compute the model frequency and mean duration of defaults, following equations (1) and (2). Last, we repeat the above procedure 100 times to obtain a simulated model stationary distribution for the duration of default, which is then compared to the data.

Figure 7 depicts the sample and simulated distributions of the duration of default for the period 1951-1980, and Figure 8 for 1981-2010. Two observations emerge. First, like in the data, the simulated distributions from the calibrated model look *exponential* – skewed to the left and with a long tail to the right. Second, like in the data, the simulated average duration of default is more than two time longer over the period after 1980 than before. The sample moments is shown in Table 4.

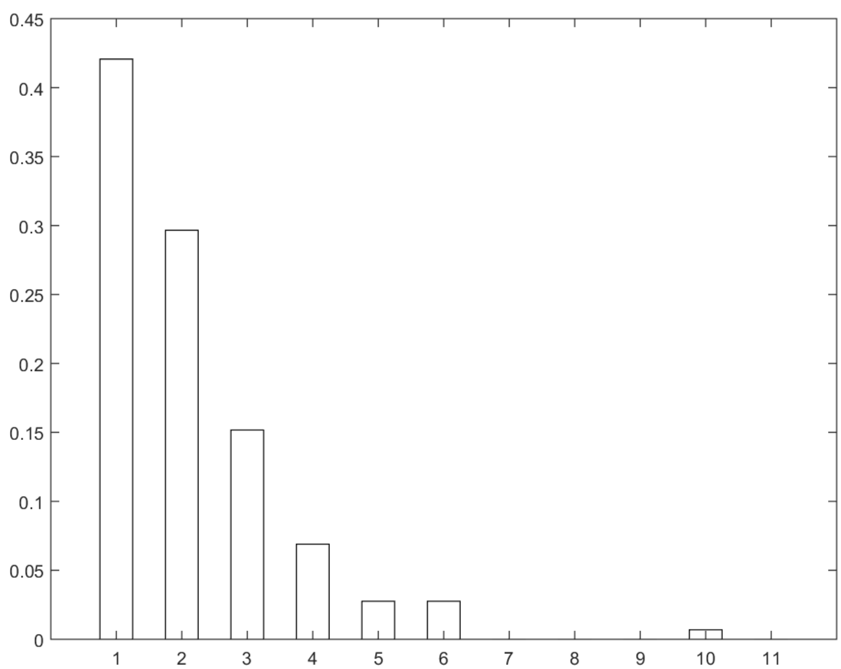
Table 4: Frequency and average duration of sovereign defaults in 12 countries

Period	Probability of default per year		Average duration per default episode	
	data	model	data	model
1951 - 1980	0.072	0.174	1.96	2.12
1981 - 2010	0.058	0.057	4.62	4.78
experiment 1		0.056		4.24
experiment 2		0.164		2.59

Note: The simulated mean of frequency and average duration is the mean of those calculated in each simulation. In experiment 1 we suppose the autarkic outputs θ_{i0} were fixed at their 1951-1980 values and compute the results over 1981-2010. In experiment 2 we suppose the parameters π, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ were fixed at their 1951-1980 values, and compute the results over 1981-2010.

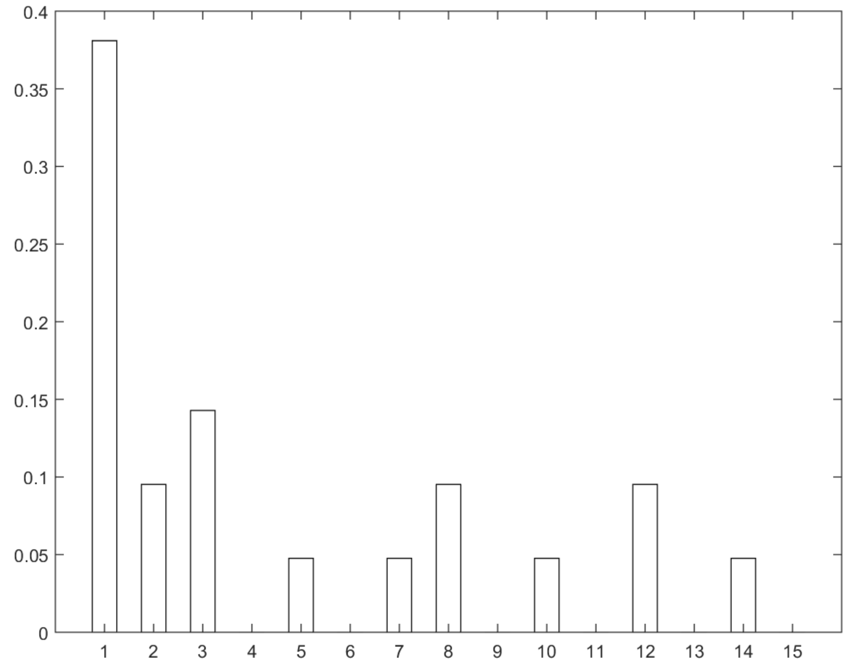


(a) The distribution of duration in 1951-1980, data.

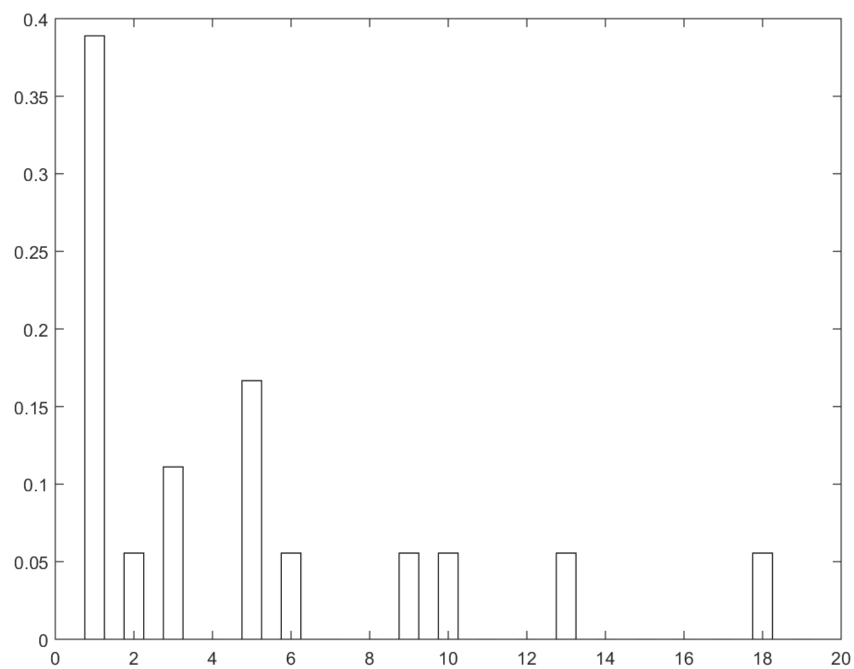


(b) The distribution of duration in 1951-1980, model.

Figure 7: Data versus Model, 1981-2010



(a) The distribution of duration in 1981-2010, data.



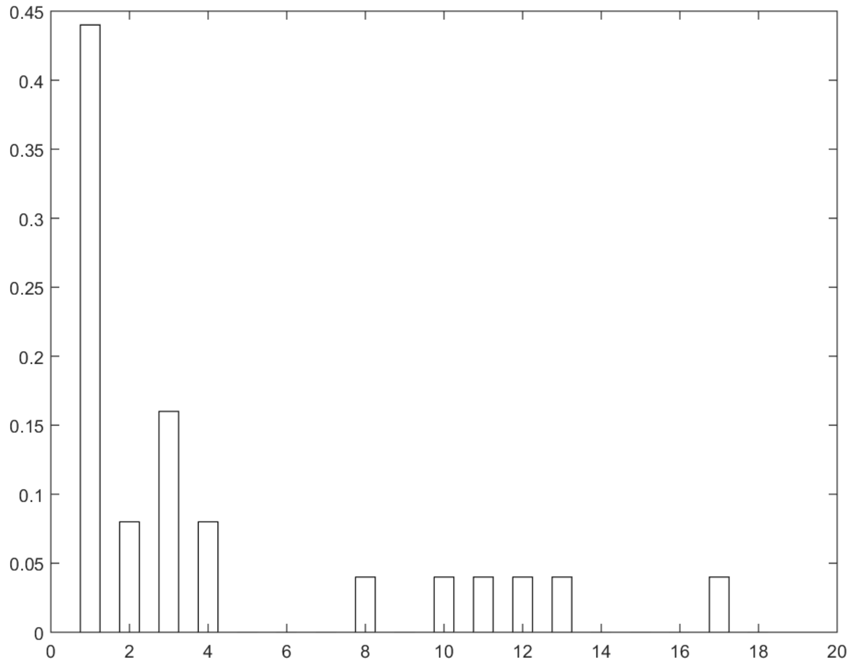
(b) The distribution of duration in 1981-2010, model.

Figure 8: Data versus Model, 1981-2010

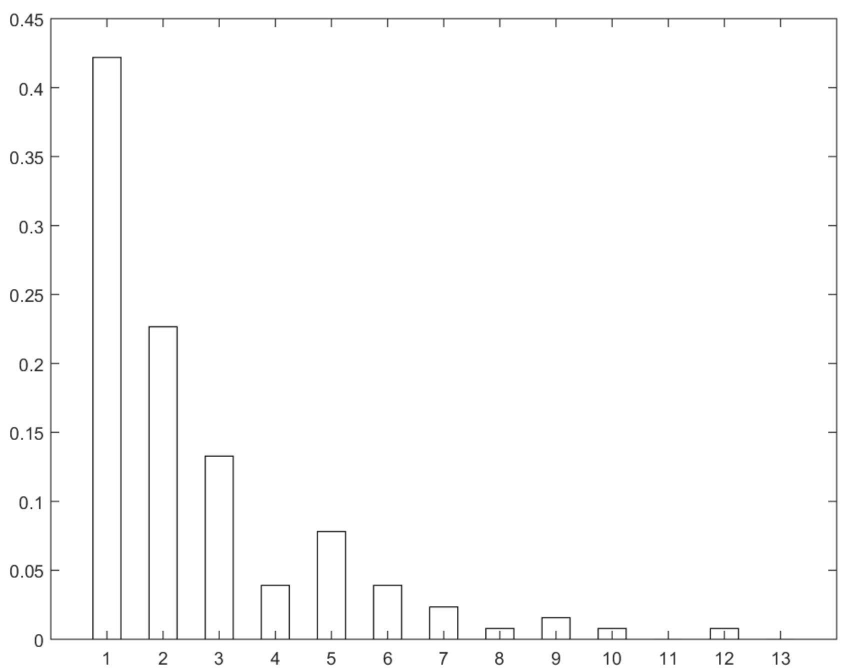
What caused the significantly longer durations after than before 1980 in the calibrated model, as in the data? The above analysis suggests two possible explanations. The first is that the sovereign country's autarkic output θ_{0i} has changed to be significantly higher after 1980 (see Table 1). This could have affected the duration, but may not be in the right direction. (i) A larger θ_0 implies a larger $\theta_0(\tilde{\theta}_2 - \tilde{\theta}_1)$, requiring stronger incentives for truth-telling, implying deeper punishment for reporting low output, and longer durations. (ii) A larger θ_0 on the other hand also allows the country to get out of a given punishment sooner – it is able to repay the lender more during default.

The second potential explanation is that the technologies that international lending supports have become more efficient, reflected in the increased (average) values of $\tilde{\theta}_i$. The lower probability that the newer technology entails for the low output induced the optimal contract to impose on the sovereign country, upon a realization of low output, larger punishments which, in turn, put the sovereign country deeper in debt once a default occurs, resulting in longer durations of default.

To identify quantitatively the effects that have caused the longer durations, we do the following counter-factual experiments. In the first experiment, we suppose the countries' autarkic output θ_{0i} were fixed at their 1951-1980 values over the period 1981-2010. Figure 9a depicts the computed distribution of durations over 1981-2010. In the second experiment, we compute the distribution of default durations over 1980-2010 supposing that the parameters π, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ were fixed at their 1951-1980 values. The outcomes are given in Figure 9b and Table 4. Clearly, it is the changes in π, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ that have resulted the longer average duration after 1980.



(a) The distribution of duration in 1981-2010, counter-factual 1.



(b) The distribution of duration in 1981-2010, counter-factual 2.

Figure 9: Two counter-factual examples for period 1981-2010

6 Conclusion

A remarkably simple model of dynamic contracting with private information and limited commitment has been constructed to study sovereign lending and default. The optimal contract generates long-run dynamics where cycles of lending and suspension alternate, resembling the observed serial default on sovereign debt. The model is calibrated to international data to account for serval stylized facts on sovereign default, including (i) observed default spells are lengthy on average, with large variability in duration, both across countries and over time; (ii) the observed distribution of default durations is highly skewed and looks exponential; and (iii) the average duration of default is more than two times longer in the thirty years after 1980 than before.

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A Appendix

A.1 The Concavity of the Value Functions

See the appendix in Luo and Wang (2015) for proving that the value functions $U(\cdot), U_s(\cdot), U_l(\cdot)$ are concave and their domains are $[V_{\min}, V_{\max})$, $[V_{\min}, V_{\max})$ and $[\widetilde{V}, V_{\max})$ respectively. In addition, conditional on lending with promised utility V_l, the incentive constraint (4) binds at the optimum but (5) does not, and it holds at the equilibrium that $m_2(V_l) \geq m_1(V_l)$, $V_2(V_l) \geq V_1(V_l)$.

A.2 Lemma 1 and proof

As in in Luo and Wang (2015), we first show that over an interval of sufficiently low Vs, the value functions $U(\cdot)$ and $U_s(\cdot)$ are upward-sloping.

Lemma 1. *Suppose $V_{\min} < \widetilde{V}$. With the optimal contract, (i) there exists $V^u \in (V_{\min}, \widetilde{V})$ such that $U(\cdot)$ is upward-sloping on the interval $[V_{\min}, V^u]$; and (ii) there exists V_s^u, with $V_s^u > V_{\min}$, such that $U_s(\cdot)$ is upward-sloping on the interval $[V_{\min}, V_s^u]$.*

Proof. (i) Suppose, with the optimal contract, $U'(V_{\min}) \leq 0$. But $U(\cdot)$ is concave, so

$$U(V_{\min}) \geq U(V),\ \forall V \in [V_{\min}, V_{\max}). \tag{20}$$

Because $V_{\min}$ is the minimum expected utility feasible for the borrower, and $V_{\min} < \widetilde{V}$, it must hold that $I(V_{\min}) = 1$. So

$$U(V_{\min}) = U_s(V_{\min}) = m_0(V_{\min}) + \beta U(V_0(V_{\min})).$$

Now consider a tuple $\{I(V), m_i(V), V_i(V)\}_{i=0,1,2}$ given by

$$I(V) = 0, m_0(V) = \theta_0 - \underline{c}, V_0(V) = V_{\min}, m_1(V) = m_2(V) = \theta_1 - \underline{c},$$

$$V_1(V) = V_2(V) = V_{\min}.$$

It is feasible at $V = \widetilde{V}$ and gives the lender a value of

$$\begin{aligned} \theta_1 - \underline{c} + \beta U(V_{\min}) &> \theta_0 - \underline{c} + \beta U(V_{\min}) \\ &\geq m_0(V_{\min}) + \beta U(V_0(V_{\min})) \\ &= U(V_{\min}), \end{aligned}$$

which implies $U(\widetilde{V}) > U(V_{\min})$, a contradiction to (20). Thus it must hold that $U'(V_{\min}) > 0$, and given that $U(\cdot)$ is concave, there exists some $V^u \in (V_{\min}, \widetilde{V})$ such that $U'(V) > 0$ for all $V \in [V_{\min}, V^u]$.

(ii) From the above proof, $U(\cdot)$ is upward-sloping on the interval $[V_{\min}, V^u]$, so

$$U(V) > U(V_{\min}) = U_s(V_{\min}),\ \forall V \in (V_{\min}, V^u]. \tag{21}$$

Now suppose, with the optimal contract, $U_s'(V_{\min}) \leq 0$. Then given that $U_s(\cdot)$ is concave, it holds that

$$U_s(V_{\min}) \geq U_s(V),\ \forall V \in [V_{\min}, V_{\max}). \tag{22}$$

Then (21) and (22) imply

$$U(V) > U_s(V), \forall V \in (V_{\min}, V^u],$$

Together with $V^u < \widetilde{V}$, we have

$$I(V) \in (0,1), \forall V \in (V_{\min}, V^u].$$

Thus $U(\cdot)$ is linear over the interval $[V_{\min}, \widetilde{V}]$, and it is also upward-sloping.

Next, consider a pair $\{m_0, V_0\}$ which is feasible to the problem **PS** at $V = u(\underline{c}) + \beta\widetilde{V}$:

$$m_0 = \theta_0 - \underline{c}, V_0 = \widetilde{V}.$$

Then

$$\begin{aligned} U_s(V) &\geq \theta_0 - \underline{c} + \beta U(\widetilde{V}) \\ &> \theta_0 - \underline{c} + \beta U(V_{\min}) \\ &\geq U_s(V_{\min}), \end{aligned}$$

where the second inequality is from assumption $V_{\min} < \widetilde{V}$ and that $U(\cdot)$ is upward-sloping over $[V_{\min}, \widetilde{V}]$. But this contradicts with (22), so $U_s'(V_{\min}) > 0$. Then given $U_s(\cdot)$ is concave, there exists some $V_s^u \in (V_{\min}, \widetilde{V})$ such that $U_s'(V) > 0$ for all $V \in [V_{\min}, V_s^u]$. □

A.3 Lemma 2 and proof

Lemma 2. *For all $V > V_{\min}$, the optimal contract involves lending in at least some periods.*

Proof. Fix $V \in (V_{\min}, V_{\max})$. Suppose the optimal contract that attains V dictates suspension in all periods. Then

$$V = \frac{u(\theta_0 - m_0)}{1-\beta}, \text{ and } U(V) = \frac{m_0}{1-\beta},$$

where $m_0 < \theta_0 - \underline{c}$ is the constant repayments from the borrower in any period.

Now consider another plan in which there is no lending until period T, but then lending occurs in all periods after T. And the lender requires the same constant repayment m_0 for all periods and all output states, except for period 0, in which he requires a repayment of m. With this plan, the expected utility of the borrower is

$$V'(m,T) = u(\theta_0 - m) + \sum_{s=1}^{T} \beta^s u(\theta_0 - m_0) + \sum_{\tau=T+1}^{\infty} \beta^\tau \sum_{i=1}^{2} \pi_i u(\theta_i - m_0).$$

Note that $V'(m,T)$ is decreasing in m and T, with $V = V'(m_0, \infty)$. Given that $\theta_0 < \theta_1 < \theta_2$ and $0 < \beta < 1$, there exist some finite T^* and $m^* \in (m_0, \theta_0)$ such that

$$V'(m^*, T^*) = V.$$

Therefore we have shown that the new plan with $\{m^*, T^*\}$ makes the borrower indifferent but gives the lender a strictly higher utility ($m^* > m_0$), a contradiction. $\square$

A.4 Suspension

As in in Luo and Wang (2015), we characterize the (unique) solution to problem **PS** in a set of first order conditions. Specifically, let α_s, μ_0, $\beta\kappa_0$ be the Lagrangian multipliers associated with constraints (10)-(12) respectively. Then

$$m_0: \quad 1 - \alpha_s u'(\theta_0 - m_0) - \mu_0 = 0, \tag{23}$$

$$\mu_0(\theta_0 - \underline{c} - m_0) = 0, \quad \mu_0 \geq 0, \tag{24}$$

$$V_0: \quad \beta U'(V_0) + \alpha_s \beta + \beta\kappa_0 = 0, \tag{25}$$

$$\kappa_0(V_0 - V_{\min}) = 0, \quad \kappa_0 \geq 0.$$

The envelope theorem gives

$$V_s: \quad U_s'(V_s) = -\alpha_s. \tag{26}$$

From (25) and (26) we have

$$U_s'(V_s) = U'(V_0) + \kappa_0.$$

Lemma 3. *For all $V_s \in (V_{\min}, V_{\max})$, $\kappa_0 = 0$ and thus*

$$U_s'(V_s) = U'(V_0). \tag{27}$$

Proof. Suppose for some $V_s > V_{\min}$, $\kappa_0 > 0$, so $V_0 = V_{\min}$. Then $U'(V_0) = U'(V_{\min}) > 0$ which, together with (25), implies

$$\alpha_s = -U'(V_0) - \kappa_0 < 0.$$

From (23) we have $\mu_0 > 0$. Thus from (24), $m_0 = \theta_0 - \underline{c}$. This contradicts with the promise-keeping constraints (10). So for all $V_s > V_{\min}$, $\kappa_0 = 0$, and equation (27) then follows immediately. □

Lemma 4. *The following holds with the optimal contract: (i) Let $V_s \in (V_{\min}, V_{\max})$ and suppose $U_s'(V_s) \geq 0$. Then $V_0(V_s) = (V_s - u(\underline{c}))/\beta$, $m_0(V_s) = \theta_0 - \underline{c}$. (ii) Let $V_s \in (V_{\min}, V_{\max})$ and suppose $U'(V_s) \geq 0$. Then $V_0(V_s) > V_s$.*

Proof. (i) Let $V_s \in (V_{\min}, V_{\max})$ and let $U_s'(V_s) \geq 0$. From (26), $-\alpha_s = U_s'(V_s) \geq 0$. And from (23) we have

$$\mu_0 = 1 - \alpha_s u'(\theta_0 - m_0) > 0.$$

Then (24) implies $m_0(V_s) = \theta_0 - \underline{c}$ and from (10) we have

$$V_0(V_s) = \frac{V_s - u\left(\theta_0 - m_0(V_s)\right)}{\beta} = \frac{V_s - u(\underline{c})}{\beta}.$$

(ii) Now suppose $U'(V_s) \geq 0$ and $V_0(V_s) \leq V_s$. From Lemma 3 we have

$$U_s'(V_s) = U'(V_0(V_s)) \geq U'(V_s) \geq 0.$$

But if $U_s'(V_s) \geq 0$, then we know from (i) that $V_0(V_s) = (V_s - u(\underline{c}))/\beta > V_s$. A contradiction, and the proof is done. □

A.5 Lending

As in Luo and Wang (2015), we restate problem **PL** as

$$U_l(V_l) = \max_{\{m_i, V_i\}_{i=1,2}} \sum_{i=1}^{2} \pi_i \left(m_i + \beta U(V_i)\right)$$

subject to

$$\alpha_l : \quad \textstyle\sum_{i=1}^{2} \pi_i \left(u(\theta_i - m_i) + \beta V_i\right) = V_l, \tag{28}$$

$$\mu_i : \quad m_i \leq \theta_i - \underline{c}, \; i = 1, 2, \tag{29}$$

$$\beta \pi_i \kappa_i : \quad V_i \geq V_{\min}, \; i = 1, 2. \tag{30}$$

$$\gamma_2 : \quad u(\theta_2 - m_2) + \beta V_2 = u(\theta_2 - m_1) + \beta V_1, \tag{31}$$

The solution to this problem can then be characterized in the following first order conditions:

$$m_1 : \quad \pi_1 - \alpha_l \pi_1 u'(\theta_1 - m_1) + \gamma_2 u'(\theta_2 - m_1) - \mu_1 = 0, \tag{32}$$

$$\mu_1(\theta_1 - \underline{c} - m_1) = 0, \quad \mu_1 \geq 0, \tag{33}$$

$$m_2 : \quad \pi_2 - \alpha_l \pi_2 u'(\theta_2 - m_2) - \gamma_2 u'(\theta_2 - m_2) - \mu_2 = 0, \tag{34}$$

$$\mu_2(\theta_2 - \underline{c} - m_2) = 0, \quad \mu_2 \geq 0, \tag{35}$$

$$V_1 : \quad \pi_1 \beta U'(V_1) + \alpha_l \pi_1 \beta - \gamma_2 \beta + \beta \pi_1 \kappa_1 = 0, \tag{36}$$

$$\kappa_1(V_1 - V_{\min}) = 0, \quad \kappa_1 \geq 0,$$

$$V_2 : \quad \pi_2 \beta U'(V_2) + \alpha_l \pi_2 \beta + \gamma_2 \beta + \beta \pi_2 \kappa_2 = 0, \tag{37}$$

$$\kappa_2(V_2 - V_{\min}) = 0, \quad \kappa_2 \geq 0.$$

And the envelope theorem gives

$$\forall V_l : \quad U_l'(V_l) = -\alpha_l. \tag{38}$$

Combining (36), (37) and (38) we have

$$U_l'(V_l) = \pi_1 U'(V_1) + \pi_2 U'(V_2) + \pi_1 \kappa_1 + \pi_2 \kappa_2. \tag{39}$$

Lemma 5. *With the optimal contract,*
(i) $\kappa_1 = 0$*, for all* $V_l \in (\widetilde{V}, V_{\max})$ *such that* $U_l'(V) \geq 0$*; and*
(ii) $\kappa_2 = 0$*, for all* $V_l \in [\widetilde{V}, V_{\max})$.

Proof. The incentive constraints imply $V_2 \geq V_1$ which, together with $V_1 \geq V_{\min}$, implies $V_2 \geq V_{\min}$, or $\kappa_2 = 0$ for all $V_l \in [\widetilde{V}, V_{\max})$, and this proves (ii).

To prove (i), suppose $V_l \in (\widetilde{V}, V_{\max})$ and $U_l'(V_l) \geq 0$ but $\kappa_1 > 0$. Then $V_1 = V_{\min}$ and $U'(V_1) = U'(V_{\min}) > 0$. From (38) we know $\alpha_l = -U_l'(V_l) \leq 0$, thus from (32) we must have

$\mu_1 > 0$ and then $m_1 = \theta_1 - \underline{c}$. But constraints (28) and (31) imply

$$\sum_{i=1}^{2} \pi_i u(\theta_i - m_1) + \beta V_1 = V_l, \tag{40}$$

or

$$V_l = \sum_{i=1}^{2} \pi_i u(\theta_i - \theta_1) + \beta V_{\min} = \widetilde{V},$$

contradicting with $V_l > \widetilde{V}$. This proves (i). $\square$

A.6 Proof of part (i) of Theorem 1

The proof is divided into three parts, in Lemmas 6 – 8. Suppose

$$0 \leq V_{\min} \equiv \frac{u(\theta_0)}{1-\beta} < \frac{\beta[\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})]}{1-\beta^2}.$$

Remember that

$$\widehat{V} = u(\underline{c}) + \beta \widetilde{V}.$$

Lemma 6. *The following holds with the optimal contract:*
(i) $U_s'(V) = U'(V_{\min})$ *for all* $V \in [V_{\min}, \widehat{V})$*;* $U_s'(V) < U'(V_{\min})$ *for all* $V \in (\widehat{V}, V_{\max})$*.*
(ii) $U'(V) = U'(V_{\min})$ *for all* $V \in [V_{\min}, \widetilde{V})$*;* $U'(V) < U'(V_{\min})$ *for all* $V \in (\widetilde{V}, V_{\max})$*.*
(iii) $I(V) = 1$ *for all* $V \in [V_{\min}, \widehat{V}]$*;* $I(V) \in (0,1)$*, for all* $V \in (\widehat{V}, \widetilde{V})$*, and* $I(\widetilde{V}) = 0$*.*

Proof. From Lemma 1 we have $U'(V_{\min}) > 0$. Let

$$\underline{V} = \sup\{V : U_s'(V) = U'(V_{\min}), V \in \Sigma\}, \tag{41}$$

$$\underline{V}' = \sup\{V : U'(V) = U'(V_{\min}), V \in \Sigma\}. \tag{42}$$

$\underline{V}$ and $\underline{V}'$ are finite since $U(\cdot), U_s(\cdot)$ are concave and

$$\lim_{V \to V_{\max}} U_s(V) = -\infty < U_s(V_{\min}),$$

$$\lim_{V \to V_{\max}} U(V) = -\infty < U(V_{\min}).$$

Step 1 We show $\underline{V} > V_{\min}$. From the proof of Lemma 1, we have $I(V_{\min}) = 1$. Let

$$\delta = \sup\{V' : I(V) = 1, \forall V \in [V_{\min}, V']\}.$$

Then $\delta \geq V_{\min}$. We consider two cases respectively: $\delta =< V_{\min}$ and $\delta > V_{\min}$.

Case 1: $\delta = V_{\min}$. In this case, for any $\phi \in (\delta, \widetilde{V})$, we have $I(\phi) \in (0,1)$, since the domain of $U_l(\cdot)$ is $[\widetilde{V}, V_{\max})$. This implies there exists some ϕ^1, ϕ^2 such that

$$V_{\min} \le \phi^1 < \phi < \widetilde{V} \le \phi^2,$$

$$\phi = I(\phi)\phi^1 + (1 - I(\phi))\phi^2,$$
$$U(\phi) = I(\phi)U_s(\phi^1) + (1 - I(\phi))U_l(\phi^2).$$

And $U(\cdot)$ is linear between ϕ^1 and ϕ^2, or

$$U(V) = K_\phi V + C_\phi, \forall V \in [\phi^1, \phi^2],$$

where K_ϕ and C_ϕ are constants.

Now since for any V, $U(V)$ is the unique maximum value, it's clear that $U(\cdot)$ is linear on the interval $(\delta, \widetilde{V})$, or

$$U'(V) \text{ is constant over } (\delta, \widetilde{V}). \tag{43}$$

From Lemmas 1 and 4 we have for all $\varepsilon \in (V_{\min}, V_s^u)$,

$$U_s'(\varepsilon) = U'(\varepsilon/\beta) = U'(V_{\min}).$$

The last equality is from (43). So $\underline{V} \ge V_s^u > V_{\min}$.

Case 2: $\delta > V_{\min}$. In this case, for any $w < v \in [V_{\min}, \min(\delta, V_s^u))$ we have

$$U_s(v) = U(v) = \theta_0 + \beta U(\frac{v}{\beta}),$$

$$U_s(w) = U(w) = \theta_0 + \beta U(\frac{w}{\beta}).$$

Thus, given that $U(\cdot)$ is concave, we have

$$U'(v) \le \frac{U(v) - U(w)}{v - w} = \frac{U(v/\beta) - U(w/\beta)}{v/\beta - w/\beta} \le U'(w/\beta),$$

for any w, v in the interval $[V_{\min}, \min(\delta, V_s^u))$ with $w < v$. Now consider w, v with $v \in (w, w/\beta)$. By the concavity of $U(\cdot)$, we have $U'(v) \ge U'(w/\beta)$. Therefore,

$$U'(w) = U'(v/\beta), \ \forall w, v \in [V_{\min}, \min(\delta, V_s^u)) \text{ with } v \in (w, w/\beta).$$

This then implies, immediately, that $U'(\cdot)$ is constant over the whole interval $[V_{\min}, \min(\delta, V_s^u))$.

And it then follows that $\underline{V} \geq \min\{\delta, V_s^u\} > V_{\min}$.

Given that $U_s(\cdot)$ and $U(\cdot)$ are concave, given $U(V_{\min}) = U_s(V_{\min})$ and $U(V) \geq U_s(V)$ for all $V \in \Sigma$, $\underline{V} > V_{\min}$ then implies

$$\{V : U_s'(V) = U'(V_{\min}), V \in \Sigma\} = [V_{\min}, \underline{V}].$$

Step 2 We prove $\underline{V}' = (\underline{V} - u(\underline{c}))/\beta$. Given $U_s'(V) = U'(V_{\min}) > 0$ for all $V \in [V_{\min}, \underline{V})$, we have

$$\lim_{V \to \underline{V}-} V_0(V) = \lim_{V \to \underline{V}-} (V - u(\underline{c}))/\beta = (\underline{V} - u(\underline{c}))/\beta.$$

From Lemma 3,

$$\lim_{V \to \underline{V}-} U'(V_0(V)) = \lim_{V \to \underline{V}-} U_s'(V) = U'(V_{\min}),$$

so $U'((\underline{V} - u(\underline{c}))/\beta) = U'(V_{\min})$ and then $\underline{V}' \geq (\underline{V} - u(\underline{c}))/\beta$.

Suppose $\underline{V}' > (\underline{V} - u(\underline{c}))/\beta$. There exists some $\varepsilon > 0$ such that $(\underline{V} + \varepsilon - u(\underline{c}))/\beta \leq \underline{V}'$. From the promise keeping constraint (10),

$$\begin{aligned} V_0(\underline{V} + \varepsilon) &= (\underline{V} + \varepsilon)/\beta - u\left(\theta_0 - m_0(\underline{V} + \varepsilon)\right)/\beta \\ &\leq (\underline{V} + \varepsilon)/\beta - u(\underline{c})/\beta \\ &\leq \underline{V}'. \end{aligned}$$

From (42), we have

$$U_s'(\underline{V} + \varepsilon) = U'(V_0(\underline{V} + \varepsilon)) = U'(V_{\min}).$$

This contradicts with the (41). So we have $\underline{V}' = (\underline{V} - u(\underline{c}))/\beta$.

Step 3 We show that $\underline{V}' = \widetilde{V}$ and $I(V) \in (0, 1)$ for all $V \in (\underline{V}, \underline{V}')$.

Since

$$U(V_{\min}) = U_s(V_{\min}),$$

$$U'(V) = U'(V_{\min}) = U_s'(V), \forall V \in [V_{\min}, \underline{V}).$$

So

$$U(\underline{V}) = U_s(\underline{V}).$$

And for all $V \in (\underline{V}, \underline{V}')$,

$$U'(V) = U'(V_{\min}) > U_s'(V).$$

Thus

$$U(V) > U_s(V), \forall V \in (\underline{V}, \underline{V}'], \tag{44}$$

and so

$$I(V) < 1, \forall V \in (\underline{V}, \underline{V}'].$$

Now if $I(\underline{V}') > 0$, then there exists some $V^1 < \underline{V}' < V^2$ such that

$$U'(\underline{V}') = U'(V^1) = U'(V^2) = U'(V_{\min}).$$

But this contradicts with (42). Thus $I(\underline{V}') = 0$ and $U(\underline{V}') = U_l(\underline{V}')$, and thus $\underline{V}' \geq \widetilde{V}$.

Suppose $\underline{V}' > \widetilde{V}$. Since $U_l(\cdot)$ is concave and

$$U'(V) = U'(V_{\min}), \forall V \in (\widetilde{V}, \underline{V}'),$$

$$U(V) \geq U_l(V), \forall V \in (\widetilde{V}, \underline{V}'),$$
$$U(\underline{V}') = U_l(\underline{V}'),$$

we have

$$U_l'(V) \geq U'(V_{\min}) > 0, \forall V \in (\widetilde{V}, \underline{V}').$$

Thus from Lemma 5 we have $\kappa_1 = \kappa_2 = 0$ for all $V \in (\widetilde{V}, \underline{V}')$, according to (39),

$$U_l'(V) = \pi_1 U'(V_1) + \pi_2 U'(V_2) \leq U'(V_{\min}).$$

So

$$U_l'(V) = U'(V_{\min}), \forall V \in (\widetilde{V}, \underline{V}').$$

Now consider the solution to the problem **PL** at $V \in (\widetilde{V}, \underline{V}')$:

$$\begin{aligned}
& U_l'(V) = U'(V_{\min}) \\
\Rightarrow\quad & U'(V_1(V)) = U'(V_2(V)) = U_l'(V) = U'(V_{\min}) > 0 \\
\Rightarrow\quad & \gamma_2(V) = 0, \alpha_l(V) < 0 \\
\Rightarrow\quad & \mu_1(V) > 0, \mu_2(V) > 0 \\
\Rightarrow\quad & m_1(V) = \theta_1 - \underline{c}, m_2(V) = \theta_2 - \underline{c}, \\
& V_2(V) = (V + \pi_1 u(\theta_2 - \theta_1 + \underline{c}) - (1 + \pi_1) u(\underline{c}))/\beta > V.
\end{aligned}$$

The second line follows from (39) and the fact that $U'(V) \leq U'(V_{\min}), \forall V \in [V_{\min}, V_{\max})$. The third line follows from (36) and (38). The fourth line follows from (32) and (34). The fifth line follows from (28), (33), (35) and (40).

Thus there exists some $\varepsilon > 0$ such that

$$V_2(\underline{V}' - \varepsilon) = \frac{\underline{V}' - \varepsilon + \pi_1 u(\theta_2 - \theta_1 + \underline{c}) - (1+\pi_1)u(\underline{c})}{\beta} > \underline{V}',$$

and

$$U'(V_2(\underline{V}' - \varepsilon)) = U'(V_{\min}),$$

but this contradicts with (42). So $\underline{V}' = \widetilde{V}$.

Step 4 From Step 3 we have $\underline{V}' = \widetilde{V}$, from Step 2 we have $\underline{V} = u(\underline{c}) + \beta\underline{V}' = u(\underline{c}) + \beta\widetilde{V}$. From Step 3 we also have

$$I(V) \in (0,1), \forall V \in (u(\underline{c}) + \beta\widetilde{V}, \widetilde{V}), \text{ and } I(\widetilde{V}) = 0.$$

These, together with (44) imply

$$U(\widetilde{V}) = U_l(\widetilde{V}) > U_s(\widetilde{V}).$$

This completes the proof of the lemma. □

Lemma 7. *With the optimal contract, for all $V \in [\widetilde{V}, V_{\max})$, $U_l(V) > U_s(V)$ and $I(V) = 0$.*

Proof. From Lemma 6, we have $U(V) > U_s(V)$, for all $V \in (\widehat{V}, \widetilde{V})$. Let V^* attain the maximum value for the lender, or $U'(V^*) = 0$. We show $U(V) > U_s(V)$ for all $V \in [\widetilde{V}, V_{\max})$ in two steps.

Step 1 For all $V \in [\widetilde{V}, V^*]$, we have $U'(V) \geq 0$. From Lemmas 3 and 4 we have

$$U_s'(V) = U'(V_0) = U'((V - u(\underline{c}))/\beta) \leq U'(V).$$

That is, as V increases, $U_s(V)$ increases more slowly than $U(V)$. So $U(V) > U_s(V)$ for all $V \in [\widetilde{V}, V^*]$.

Step 2 For all $V \in (V^*, V_{\max})$, we have $U'(V) < 0$. Suppose $U_s(V) = m_0 + \beta U(V_0)$. Now consider a plan at V at which lending occurs, and with $\{m_1 = m_2 = m_0, V_1 = V_2 = V_0\}$. This plan will give the lender a value at V equal to $m_0 + \beta U(V_0) = U_s(V)$, while giving the borrower

$$V' = \sum_{i=1}^{2} \pi_i u(\theta_i - m_0) + \beta V_0 > V.$$

Obviously then

$$U(V) > U(V') \geq m_0 + \beta U(V_0) = U_s(V).$$

Now for all $V \in [\widetilde{V}, V_{\max})$, we have $U(V) > U_s(V)$ which implies $I(V) < 1$. Suppose there exists $\phi \in [\widetilde{V}, V_{\max})$ such that $I(\phi) \in (0,1)$. Then there must exist ϕ^1, ϕ^2 with

$$V_{\min} \leq \phi^1 < \widetilde{V} \leq \phi < \phi^2$$

such that $U(\cdot)$ is linear between ϕ^1 and ϕ^2. This contradicts with part (ii) of Lemma 6. So $I(V) = 0$ for all $V \in [\widetilde{V}, V_{\max})$. Thus $U_l(V) = U(V) > U_s(V)$, for all $V \in [\widetilde{V}, V_{\max})$. $\square$

Lemma 8. *With the optimal contract, it holds for all* $V \in [\widetilde{V}, V_{\max})$ *that* $V_1(V) < V < V_2(V)$ *and* $m_1(V) < m_2(V)$.

Proof. We need only prove $V_1(V) < V < V_2(V)$ which, given (31), necessarily implies $m_1 < m_2$.

Fix $V \in [\widetilde{V}, V_{\max})$. Note that we have $V_1 \leq V_2$. From Lemma 7 we have $U(V) = U_l(V)$. These, together with (38) imply

$$U'(V) = U_l'(V) = -\alpha_l.$$

From (39) and Lemma 5 we have

$$\pi_1 U'(V_1) + \pi_2 U'(V_2) + \pi_1 \kappa_1 = U'(V).$$

Suppose $\kappa_1 > 0$. Then $V_1 = V_{\min}$ and

$$U'(V_2) = U'(V) + \frac{\pi_1}{\pi_2}(U'(V) - U'(V_1) - \kappa_1) < U'(V).$$

Given $U'(V) \leq U'(V_1) = U'(V_{\min})$, we have $V_2 > V \geq \widetilde{V} > V_1 = V_{\min}$.

Suppose $\kappa_1 = 0$. Then

$$\pi_1 U'(V_1) + \pi_2 U'(V_2) = U'(V) = -\alpha_l.$$

Given $V_1 \leq V_2$ and that $U(\cdot)$ is concave, we have either

$$U'(V_1) > U'(V) > U'(V_2),$$

which implies $V_1 < V < V_2$; or

$$U'(V_1) = U'(V) = U'(V_2) = -\alpha_l. \tag{45}$$

Now suppose $U'(V_1) = U'(V) = U'(V_2) = -\alpha_l$. Let

$$V' = \inf\{\omega : U'(\omega) = -\alpha_l, \omega \in [\widetilde{V}, V_{\max})\}, \tag{46}$$

and

$$V'' = \sup\{\omega : U'(\omega) = -\alpha_l, \omega \in [\widetilde{V}, V_{\max})\}. \tag{47}$$

It is clear that $\widetilde{V} \leq V' \leq V \leq V''$. Let $\{m_1', m_2', V_1', V_2', \mu_1', \mu_2', \gamma_2', \kappa_1'\}$ be the solution to the problem **PL** at V' and $\{m_1'', m_2'', V_1'', V_2'', \mu_1'', \mu_2'', \gamma_2'', \kappa_1''\}$ be the solution at V''. From (39) and Lemma 5 we have

$$\pi_1 U'(V_1') + \pi_2 U'(V_2') + \pi_1 \kappa_1' = U'(V'), \tag{48}$$

$$\pi_1 U'(V_1'') + \pi_2 U'(V_2'') + \pi_1 \kappa_1'' = U'(V'').$$

Case 1 Suppose $V_1 = V = V_2$. From (31) we have $m_2 = m_1 \leq \theta_1 - \underline{c} < \theta_2 - \underline{c}$, so $\mu_2 = 0$. From (37) we have

$$\gamma_2 = -\left(\pi_2 U'(V_2) + \alpha_l \pi_2\right) = 0.$$

This, together with (34), implies

$$\alpha_l = \frac{1}{u'(\theta_2 - m_2)}.$$

Moreover, from (32) we have

$$\alpha_l = \frac{1 - \mu_1/\pi_1}{u'(\theta_1 - m_1)}.$$

So

$$\alpha_l = \frac{1}{u'(\theta_2 - m_2)} = \frac{1 - \mu_1/\pi_1}{u'(\theta_1 - m_1)} \leq \frac{1}{u'(\theta_1 - m_1)}.$$

This contradicts with $m_1 = m_2$. Thus it is *not* optimal to have $V_2 = V = V_1$.

Case 2 Suppose $\alpha_l \leq 0$. From (32), $\gamma_2 \geq 0$ and $u'(\cdot) \geq 0$ we have $\mu_1 > 0$, so $m_1 = \theta_1 - \underline{c}$. From (40) we have

$$\begin{aligned} V_1 &= \left[V - \pi_1 u(\theta_1 - m_1) - \pi_2 u(\theta_2 - m_2)\right]/\beta \\ &= \left[V - \pi_1 u(\underline{c}) - \pi_2 u(\theta_2 - \theta_1 + \underline{c})\right]/\beta \\ &= (V - \widetilde{V} + \beta V_{\min})/\beta. \end{aligned}$$

Now suppose $V_2 \leq V$. Then $U'(V_2) \geq U'(V) = -\alpha_l \geq 0$. This, together with (37), implies

$$\alpha_l + \gamma_2/\pi_2 = -U'(V_2) \leq 0.$$

From (34) we have

$$\mu_2 = \pi_2 \left[1 - (\alpha_l + \gamma_2/\pi_2)u'(\theta_2 - m_2)\right] > 0,$$

and so $m_2 = \theta_2 - \underline{c}$. From (28) and $0 < \beta < 1$, we have

$$V_2 = \frac{V + \pi_1 u(\theta_2 - \theta_1 + \underline{c}) - (1 + \pi_1)u(\underline{c})}{\beta} > V,$$

a contradiction. So we have $V_2 > V$.

Next, suppose $V_1 = (V - \widetilde{V} + V_{\min})/\beta \geq V$. Then

$$(\widetilde{V} - V_{\min})/(1 - \beta) \leq V.$$

Given

$$U'(V_2) = U'(V_1) = U'(V) = -\alpha_l,$$

we have $V'' \geq V_2 > V$. Consider now the solution at V''. Using the same argument for proving $V_2 > V$, we can show $V_2'' > V''$. From (47), we have $U'(V_2'') < U'(V'') = -\alpha_l$. Since $U'(V'') = -\alpha_l \geq 0$, from Lemma 5 we have

$$\pi_1 U'(V_1'') + \pi_2 U'(V_2'') = U'(V'') = -\alpha_l.$$

So $U'(V_1'') > U'(V'')$. Since $U(\cdot)$ is concave, we have $V_1'' = (V'' - \widetilde{V} + V_{\min})/\beta < V''$. This implies $V'' < (\widetilde{V} - V_{\min})/(1 - \beta) \leq V$, a contradiction. So it must hold $V_1 < V < V_2$.

Case 3 Suppose $\alpha_l > 0$. The following shows that this case is not optimal. In Case 1 we have shown that it is not optimal to have $V_1 = V = V_2$. Given $V_1 \leq V_2$, there are two cases to be considered: $V_1 < V$ and $V_1 \geq V, V_2 > V$.

Case 3.1 Suppose $V_1 < V$. Then $V' \leq V_1 < V$. Now consider the solution at V'. Given $V_1' \leq V_2'$, we have $U'(V_1') \geq U'(V_2')$. There are two cases to be considered: $U'(V_1') = U'(V_2')$ and $U'(V_1') > U'(V_2')$, and we derive a contradiction for both cases.

Case 3.1.1 Suppose $U'(V_1') = U'(V_2')$. If $\kappa_1' > 0$, then $V_1' = 0$ and (48) implies

$$U'(V') = U'(V_1') + \pi_1 \kappa_1' > U'(0),$$

a contraction. So we have $\kappa_1' = 0$, and

$$U'(V_1') = U'(V') = U'(V_2') = -\alpha_l.$$

Then together with (37) we have $\gamma_2 = \gamma_2' = 0$. So (32) implies

$$-\alpha_l \pi_1 \left[u'(\theta_1 - m_1) - u'(\theta_1 - m_1') \right] = \mu_1 - \mu_1'. \tag{49}$$

If $\mu_1 = 0, \mu_1' = 0$, then because $u(\cdot)$ is strictly concave, we have $m_1 = m_1'$. If $\mu_1 = 0, \mu_1' > 0$, then $m_1' = \theta_1 - \underline{c}$. The right hand side of (49) is negative, but the left hand side of (49) is

$$-\alpha_l \pi_1 \left[u'(\theta_1 - m_1) - u'(\underline{c}) \right] \geq 0,$$

a contradiction. If $\mu_1 > 0, \mu_1' = 0$, then $m_1 = \theta_1 - \underline{c}$. The right hand side of (49) is positive, but the left hand side of (49) is

$$-\alpha_l \pi_1 \left[u'(\underline{c}) - u'(\theta_1 - m_1') \right] \leq 0,$$

a contradiction. If $\mu_1 > 0, \mu_1' > 0$, then $m_1 = m_1' = \theta_1 - \underline{c}$. These imply $m_1 = m_1'$. So, from (40) we have

$$\begin{aligned} & V' - \beta V_1' = V - \beta V_1 \\ \Rightarrow \quad & V_1' = V_1 - V/\beta + V'/\beta \\ & \quad = V' + (V' - V)(1/\beta - 1) + (V_1 - V) < V'. \end{aligned}$$

But given $U'(V_1') = U'(V') = -\alpha_l$, this contradicts with the (46).

Case 3.1.2 Suppose $U'(V_1') > U'(V_2')$. From (48) we have

$$U'(V_2') < U'(V') = -\alpha_l \leq U'(V_1'),$$

which implies $V_1' \leq V_1 \leq V_2 < V_2'$. The following shows $m_1' \geq m_1$ and $m_2' \leq m_2$.

From (37), we have $\gamma_2 = 0, \gamma_2' > 0$. So from (32), we have

$$\pi_1 = \alpha_l \pi_1 u'(\theta_1 - m_1) + \mu_1 = \alpha_l \pi_1 u'(\theta_1 - m_1') + \mu_1' - \gamma_2' u'(\theta_2 - m_1').$$

So either $u'(\theta_1 - m_1') > u'(\theta_1 - m_1)$ or $\mu_1' > \mu_1 \geq 0$. In both cases, $m_1' \geq m_1$ holds.

From (34), we have

$$\pi_2 = \alpha_l \pi_2 u'(\theta_2 - m_2) + \mu_2 = \alpha_l \pi_2 u'(\theta_2 - m_2') + \mu_2' + \gamma_2' u'(\theta_2 - m_2').$$

So either $u'(\theta_2 - m_2') < u'(\theta_2 - m_2)$ or $0 \leq \mu_2' < \mu_2$. In both cases, $m_2' \leq m_2$ holds.

But, from (31) we have

$$\begin{aligned} & V_2' - V_1' > V_2 - V_1 \\ \Rightarrow \quad & u(\theta_2 - m_1') - u(\theta_2 - m_2') > u(\theta_2 - m_1) - u(\theta_2 - m_2), \end{aligned}$$

contradicting with $m_1' \geq m_1, m_2' \leq m_2$.

Case 3.2 Suppose $V_1 \geq V$ and $V_2 > V$. These, together with (45) and (47), give $V < V_2 \leq V''$. Consider then the solution at V''. Given $V_1'' \leq V_2''$, we have $U'(V_1'') \geq U'(V_2'')$. There are two cases to be considered: $U'(V_1'') = U'(V_2'')$ and $U'(V_1'') > U'(V_2'')$, and we derive a contradiction for both cases.

Case 3.2.1 Suppose $U'(V_1'') = U'(V_2'')$. Applying the same argument for deriving the contradiction in Case 3.1.1 we can show

$$U'(V_1'') = U'(V'') = U'(V_2'') = -\alpha_l,$$

and $m_1 = m_1''$. Then from (40) we have

$$\begin{aligned} & V'' - \beta V_1'' = V - \beta V_1 \\ \Rightarrow \quad & V_1'' = V_1 - V/\beta + V''/\beta \\ & \quad = V'' + (V'' - V)(1/\beta - 1) + (V_1 - V) > V''. \end{aligned}$$

But given $U'(V_1'') = U'(V'') = -\alpha_l$, this contradicts with (47).

Case 3.2.2 Suppose $U'(V_1'') > U'(V_2'')$. Then $V_1'' \leq V_1 \leq V_2 < V_2''$. Applying the same argument for deriving the contradiction in Case 3.1.2 we will have a contradiction.

To summarize, in the case of $U'(V_1) = U'(V) = U'(V_2) = -\alpha_l$, we also have $V_1 < V < V_2$. The proof of the lemma is now complete. $\square$

A.7 Proof of part (ii) of Theorem 1

Suppose

$$\frac{\beta[\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})]}{1 - \beta^2} \leq V_{\min} < \frac{\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})}{1 - \beta}.$$

We have $V_{\min} < \widetilde{V}$. Following the proof of Lemma 1, we can show that $U'(V_{\min}) > 0$. Then following the steps in the proofs of Lemmas 6 – 8 we can show part (ii) of Theorem 1.

A.8 Proof of part *(iii)* of Theorem 1

Suppose

$$V_{\min} \geq \frac{\pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c})}{1-\beta}.$$

We have $V_{\min} \geq \widetilde{V}$. So for all $V \in [V_{\min}, V_{\max})$, the contract is free to enforce lending or suspension. There are three cases to consider.

Case 1: $U_l(V_{\min}) > U_s(V_{\min})$. In this case, following the same argument as in the proof of Lemma 7, it is straightforward to show that $U(V) = U_l(V) > U_s(V)$, for all $V \in [V_{\min}, V_{\max})$.

Case 2: $U'(V_{\min}) \leq 0$. In this case, we show $U_l(V_{\min}) > U_s(V_{\min})$, as in Case 1. Suppose, conditional on suspension, the optimal contract has, at $V = V_{\min}$,

$$V_{\min} = u(\theta_0 - m_0) + \beta V_0, \tag{50}$$

$$U_s(V_{\min}) = m_0 + \beta U(V_0).$$

Then consider a deviation from suspension to lending, with

$$\{m_1 = m_2 = m_0 + \delta_m, \quad V_1 = V_2 = V_0 - \delta_v\},$$

where $\delta_m \in [0, \theta_1 - \underline{c} - m_0], \delta_v \in [0, V_0 - V_{\min}]$ and δ_m, δ_v satisfy

$$V_{\min} = \pi_1 u(\theta_1 - m_0 - \delta_m) + \pi_2 u(\theta_2 - m_0 - \delta_m) + \beta(V_0 - \delta_v).$$

Such δ_m and δ_v must exist because of (50) and

$$V_{\min} \geq \widetilde{V} = \pi_1 u(\underline{c}) + \pi_2 u(\theta_2 - \theta_1 + \underline{c}) + \beta V_{\min}.$$

Given that $U'(V_{\min}) \leq 0$ and $U(\cdot)$ is concave, we have $U'(V) \leq 0$ for all $V \in [V_{\min}, V_{\max})$. So the deviation thus constructed gives the lender more value than he had under suspension. We therefore have $U_l(V_{\min}) > U_s(V_{\min})$.

Case 3: $U_l(V_{\min}) \leq U_s(V_{\min})$ and $U'(V_{\min}) > 0$. Using the same argument as in the proof of Lemma 6, we can also derive a contradiction in this case. This completes the proof of the theorem.